Tangent Planes and Linear Approximations

Lecture 22

February 21, 2007

Tangent Planes

Definition

- Let S be a surface with equation $z = f(x, y)$.
- Let $P(x_0, y_0, z_0)$ be a point on S.
- Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S.
- Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2.
- The **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

Equations of the tangent plane

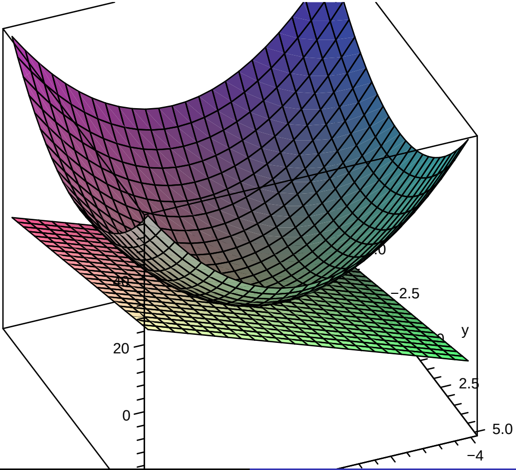
Definition

- Suppose f has a continuous partial derivatives.
- An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

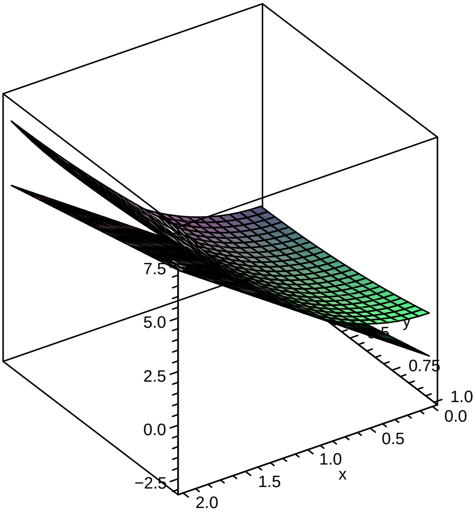
$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example

- Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.



Linear Approximations



Linear Approximations

Definition

- The linear function whose graph is this tangent plane

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

Examples

- Find the linearization of the function $f(x, y) = \sqrt{xy}$ at the point $(4, 16)$.
- Find the linearization of the function $f(x, y) = 1 + y + x \cos y$ at $P_0(0, 0)$.

The increment of z

Definition

- Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a).$$

- If f is differentiable at a, then

$$\Delta y = f^{'}(a)\Delta x + \epsilon \Delta x,$$

where $\epsilon \to 0$ as $\Delta x \to 0$.

The increment of z

Definition

- If $z = f(x, y)$ and x changes from (a, b) to $(a + \Delta x, b + \Delta y)$, then the **increment** of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

- If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Fact

- *If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).*

Example

Example

- Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there.

Differentials

Definition

- For a differentiable function $z = f(x, y)$ we define the **differential** $\mathrm{d}z$, also called the **total differential**, is defined by

$$\mathrm{d}z = f_x(x, y)\mathrm{d}x + f_y(x, y)\mathrm{d}y = \frac{\partial z}{\partial x}\mathrm{d}z + \frac{\partial z}{\partial y}\mathrm{d}y,$$

  where the **differentials** $\mathrm{d}x$ and $\mathrm{d}y$ are independent variables.
- If $\mathrm{d}x = \Delta x = x - a$ and $\mathrm{d}y = \Delta y = y - b$ the the differential of z is

$$\mathrm{d}z = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Examples

- If $f(x, y) = x^2 + 3xy - y^2$, find the differential $\mathrm{d}z$.
- If x changes from 2 to 2.05 and y changes 3 to 2.96, compare the values of Δz and $\mathrm{d}z$.