Partial Derivatives

Lecture 21

February 19, 2007

Partial derivative of f with respect to x

Definition

- Let $f(x, y)$ be a function of two variables.
- Let $y = b$ be fixed.
- Then $g(x) = f(x, b)$ is a function of a single variable x.
- If g has a derivative at a, then we call it the **partial derivative of f with respect to x at** (a, b)

$$f_x(a, b) = g'(a)$$

Partial derivative of f with respect to y

Definition

- Now keep $x = a$ fix.
- Let $h(y) = f(a, y)$.
- If h has a derivative at b, then we call it the **partial derivative of f with respect to y at (a, b)**

$$f_y(a, b) = h^{'}(b)$$

Partial Derivatives

- By the definition of a derivative, we have

$$\begin{aligned} f_x(a,b) &= \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} \\ f_y(a,b) &= \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h} \end{aligned}$$

- The partial derivatives of $f(x,y)$ are the functions $f_x(x,y)$ and $f_y(x,y)$ obtained by letting the point (a,b) vary.

Notations

- If $z = f(x, y)$, we write

$$\begin{aligned} f_x(x,y) &= f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f \\ f_y(x,y) &= f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f \end{aligned}$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

- To find f_x regard y as a constant and differentiate $f(x, y)$ with respect to x.
- To find f_y regard x as a constant and differentiate $f(x, y)$ with respect to y.

Examples

Examples

- If $f(x,y) = x^2 + 3x^3y - xy^2$ find $f_x(0,1)$ and $f_y(1,0)$
- Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the functions
  - $$f(x,y) = \frac{2y}{y + \cos x}$$
  - $$f(x,y) = e^{x^2+y^2+1}$$
  - $$f(x,y) = \ln(x+y)$$

Examples (cont'd)

Example

- Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Interpretations of Partial Derivatives

- Partial derivative can be interpreted as rates of change.
- The geometric interpretation: the partial derivatives are the slopes of the tangent lines at $P(a, b, c)$ to the curves given by the intersection of the surface given by $z = f(x, y)$ and the planes $x = a$ and $y = b$.

Higher Derivatives

Definition

- If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables.
- So why stop here?
- The **second partial derivatives** of f are

$$
\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial^2 x} = \frac{\partial^2 z}{\partial^2 x} \\
f_{xy} &= \frac{\partial^2 f}{\partial x \partial y} = \cdots \\
f_{yx} &= \frac{\partial^2 f}{\partial y \partial x} = \cdots \\
f_{yy} &= \frac{\partial^2 f}{\partial^2 y}
\end{aligned}
$$

Example

Example

- Find the second derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Clairaut's Theorem

Theorem

- *Suppose* f *is defined on a disk* D *that contains the point* (a, b). *If the functions* f_{xy} *and* f_{yx} *are both continuous on* D, *then*

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Partial derivatives of order 3 and higher

Examples

- Calculate f_{xxy} if $f(x, y) = \sin(3x^2 + xy)$.
- Find the partial derivatives of

$$f(x, y) = \int_x^y e^{t^2+t+1} \mathrm{d}t$$

- Find $f_x, f_y,, f_{xy}, f_{yx}$ for

$$f(x, y) = xye^{3xy}$$