

# Arc Length and Motion in Space

Lecture 19

February 14, 2007

## Recall: the Length of a Plane Curve

### Fact

Suppose that a plane curve has parametric equations  $x = f(t)$  and  $y = g(t)$ ,  $a \leq t \leq b$ . The **length** is given by the formula

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \end{aligned}$$

# The Length of A Space Curve

## Definition

- The **length of a space curve**  $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$  is

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt. \end{aligned}$$

- Note that

$$L = \int_a^b |\mathbf{r}'(t)| dt.$$

- - Find the length of the arc of the circular helix with vector equation  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$  from the point  $(1, 0, 0)$  to the point  $(1, 0, 2\pi)$ .
  - Find the point where you have to stop if you want to travel only half of the previous length.
- Find the length of the curve  $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \ln t\mathbf{k}$  with  $0 \leq t \leq 1$ .

## Definition

- Let  $\mathbf{r}(t)$  be a space curve. The **velocity vector**  $\mathbf{v}(t)$  at time  $t$  is

$$\mathbf{v}(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t).$$

- The **speed** of the particle at time  $t$  is the magnitude of the velocity, that is,  $|\mathbf{v}(t)|$ .
- The **acceleration** of the particle is defined as the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

## Example

- Find the velocity, acceleration, and speed of a particle with position vector  $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$ .
- Find the velocity and position vectors of a particle that has the acceleration  $\mathbf{a}(t) = -10\mathbf{k}$ , initial velocity  $\mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}$  and initial position  $\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j}$ .

## Example

A projectile is fired with an angle of elevation of  $\pi/6$  and initial velocity of 1200km/sec. Where does the projectile hit the ground and with what speed?

- Recall: **Newton's Second Law of Motion**

$$\mathbf{F}(t) = m\mathbf{a}(t).$$

- So

$$\mathbf{F}(t) = m\mathbf{a} = -mg\mathbf{j},$$

where  $g = |\mathbf{a}| \simeq 9.8m/s^2$ .