The Dot Product

January 5, 2007

The Dot Product

- If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

- It is also called the **scalar product** or **inner product**.

Examples

- $\langle 2, 1 \rangle \cdot \langle -1, 3 \rangle$.

Examples

- $\langle 2, 1 \rangle \cdot \langle -1, 3 \rangle$.

- $\langle 3, -2, 1 \rangle \cdot \langle 0, 1, 1 \rangle$.

Properties of the Dot Product

- If $\mathbf{a}, \mathrm{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

  1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
  2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
  3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
  4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
  5. $\mathbf{0} \cdot \mathbf{a} = 0.$

The angle between two vectors

- If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

The angle between two vectors

- If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

- If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Orthogonal vectors

- $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Direction Angles

- The **direction angles** of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ that $\mathbf{a}$ makes with the positive $x-$,$y-$, and $z-$axes.

- The cosines of these direction angles are called the **direction cosines** of the the vector $\mathbf{a}$:

$$\cos\alpha = \frac{a_1}{|\mathbf{a}|},\ \cos\beta = \frac{a_2}{|\mathbf{a}|},\ \cos\gamma = \frac{a_3}{|\mathbf{a}|}.$$

Projections

- Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$:

$$\text{comp}_a\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

Projections

- Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$:

$$\text{comp}_a\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

- Vector projection of $\mathbf{b}$ onto $\mathbf{a}$

$$\text{proj}_a b = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

Work done by a constant force

- The work done by a constant force $\mathbf{F}$ is

$$\mathbf{F} \cdot \mathbf{D},$$

  where $\mathbf{D}$ is the displacement vector.

Work done by a constant force

- The work done by a constant force $\mathbf{F}$ is

$$\mathbf{F} \cdot \mathbf{D},$$

  where $\mathbf{D}$ is the displacement vector.

- Example: A constant force $\mathbf{F} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ moves an object along a straight line from the point $(1, 0, 0)$ to $(-3, 2, 3)$. Find the work done.