Problems from Calculus, by Adams, due April 9. All problems, three points each.

(7) Approximate $9^{1/3}$ by using the Taylor polynomial of degree 2 for $f(x) = x^{1/3}$ around $x = 8$. Estimate the error and give an interval in which you can be sure $9^{1/3}$ lies. (Notice, this has nothing to do with the interval of convergence.)

$$\begin{array}{ll} f(x) = x^{1/3} & f(8) = 2 \\ f'(x) = \frac{1}{3}x^{-2/3} & f'(8) = \frac{2^{-2}}{3} = \frac{1}{12} \\ f''(x) = -\frac{2}{9}x^{-5/3} & f''(8) = -\frac{2 \cdot 2^{-5}}{9} = -\frac{1}{144} \\ f'''(x) = \frac{10}{27}x^{-8/3} & \end{array}$$

$$P_2(x) = 2 + \frac{1}{12}x - \frac{1}{144 \cdot 2!}x^2 = 2 + \frac{1}{12}x - \frac{1}{288}x^2$$

$$P_2(9) = 2 + \frac{9}{12} - \frac{81}{144} = \frac{35}{16}$$

In decimal form, $P_2(9) = 2.1875$.

The error is $\frac{10}{27 \cdot 3! X^{8/3}}$, where X is between 8 and 9. Note that the error is positive. Since X is in the denominator, the error term would be largest when $X = 2$, so the error is no more than $\frac{10}{27 \cdot 6 \cdot 2^8} = 5/20736$, or in decimal form .0002411265432, computations courtesy of Maple! There are many ways you might decide to take the decimal approximation, but the interval in which you know the value lies is

$$\left[\frac{35}{16}, \frac{35}{16} + \frac{5}{20736}\right] = \left[\frac{35}{16}, \frac{45365}{20736}\right] = [2.1875, 2.187741127],$$

again courtesy of Maple.

(12)

$$\begin{aligned} f(x) &= \sin x \text{ so } f(\pi/4) = \sqrt{2}/2 \\ f'(x) &= \cos x \text{ so } f'(\pi/4) = \sqrt{2}/2 \\ f''(x) &= -\sin x \text{ so } f''(\pi/4) = -\sqrt{2}/2 \\ f'''(x) &= -\cos x \text{ so } f'''(X) \geq -\sqrt{2}/2 \text{ and is negative.} \end{aligned}$$

(Thinking about the graphs of the sine and cosine functions helps here.)

$$P_2(x) = \sqrt{2}/2 + \sqrt{2}(x - \pi/4)/2 - \sqrt{2}(x - \pi/4)^2/4.$$

Now $47^o - 45^o$ is $2\pi(2/360) = \pi/90$ radians, which is our $x - \pi/4$. Thus

$$\begin{aligned} P_2(\pi/4 + \pi/90) &= \sqrt{2}/2 + \sqrt{2}(\pi/90)/2 - \sqrt{2}(\pi/90)^2/4 \\ &= \sqrt{2}/2 + \sqrt{2}\pi/180 - \sqrt{2}\pi^2/32400 \end{aligned}$$

which is about .7313586699. Since we know the absolute value of $f'''(X)$ is no more than $\sqrt{2}/2$, we know the absolute value of the remainder is no more than $\frac{\sqrt{2}}{2\cdot 3!}(\pi/90)^3 = \sqrt{2}\pi^3/8748000$, which is approximately .000005012516804, courtesy of Maple. Since the remainder is negative, the interval in which we are sure $\cos 47^o$ lies is therefore

$$[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{180} - \frac{\sqrt{2}\pi^2}{32400} - \frac{\sqrt{2}\pi^3}{8748000}, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{180} - \frac{\sqrt{2}\pi^2}{32400}].$$

In more humane terms of a decimal approximation, the interval is approximately

$$[.731354, .731359]$$

(15) Expand $\sin x$ as a Taylor polynomial of degree four around $x = \pi/4$ plus a remainder.

$$\begin{aligned} f(x) &= \sin x \text{ so } f(\pi/4) = \sqrt{2}/2 \\ f'(x) &= \cos x \text{ so } f'(\pi/4) = \sqrt{2}/2 \\ f''(x) &= -\sin x \text{ so } f''(\pi/4) = -\sqrt{2}/2 \\ f'''(x) &= -\cos x \text{ so } f'''(\pi/4) = -\sqrt{2}/2 \\ f''''(x) &= \sin x \text{ so } f''''(\pi/4) = \sqrt{2}/2 \\ f'''''(x) &= \cos x \end{aligned}$$

From this and the usual formula for a Taylor polynomial, as in problem 12, we get

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x - \frac{\pi}{4})^2 - \frac{\sqrt{2}}{12}(x - \frac{\pi}{4})^3 + \frac{\sqrt{2}}{48}(x - \frac{\pi}{4})^4 + \frac{\sin X}{120}(x - \frac{\pi}{4})^5,$$

where X is some number between x and $\frac{\pi}{4}$.

(18) Express $\tan x$ as a Taylor polynomial of degree 3 plus a remainder.

$$\begin{aligned} f(x) &= \tan x \text{ so } f(0) = 0 \\ f'(x) &= \sec^2 x = 1 + \tan^2 x \text{ so } f'(0) = 1 \\ f''(x) &= 2\tan x \sec^2 x = 2\tan x + 2\tan^3 x \text{ so } f''(0) = 0 \\ f'''(x) &= 2\sec^2 x + 6\tan^2 x \sec^2 x = 2 + 8\tan^2 x + 6\tan^4 x \text{ so } f'''(0) = 2 \\ f''''(x) &= 16\tan x \sec^2 x + 24\tan^3 x \sec^2 x \end{aligned}$$

(Notice that by always expressing things I was going to take a derivative as powers of tan I was able to avoid using the product rule.) For the remainder we will have by factoring that $f''''(X) = 8\sec^2 X \tan X(2 + 3\tan^2 X)$.

This gives us

$$\tan x = 1 + \frac{x^3}{3} + 8\sec^2 X \tan X(2 + 3\tan^2 X)\frac{x^4}{24}.$$

Unless we have some constraint on x, that value for the remainder really isn't much help, because, for example if X could be close to $\pi/2$, that remainder could be pretty big! In fact, if you get Maple to plot $\tan x$ between -1.5 and 1.5 and the Taylor polynomial, you probably won't like the approximation very much near the edges. In other words, that remainder term can get pretty big pretty fast.

(2) Find the center, radius, and open interval of convergence of

$$\sum_{n=0}^{\infty} 3n(x+1)^n.$$

The center is -1. For the radius of convergence,

$$\lim_{n\to\infty}\left|\frac{3(n+1)}{3n}\right| = \lim_{n\to\infty}(1 + \frac{1}{n}) = 1 = L.$$

Therefore $R = 1/L = 1$. Therefore the open interval of convergence is $(-2, 0)$.

(3) Find the center, radius, and open interval of convergence of

$$\sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{x+2}{2}\right)^n.$$

We write

$$\sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{x+2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n2^n}(x+2)^n.$$

From this we see that the centeer is -2 and that $a_n = \frac{1}{n2^n}$. Thus

$$L = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n2^n}}\right| = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{1}{2} = \frac{1}{2}.$$

Thus $R = 2$ and the open interval of convergence is $(-4, 0)$.

(1) We could take derivatives and find the MacLaurin series by using the definition of a MacLaurin series, but it is easier to use the fact that

$$e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

to write

$$e^{3x+1} = e\cdot e^{3x} = e(1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots) = \sum_{n=0}^{\infty} 3^n e\frac{x^n}{n!},$$

and this is the Maclaurin series. Since the series for e^x converges for every number x, so does this series.

(7) We could take derivatives and find a pattern. However $\sin x\cos x$ should remind us of an old trig identity, namely $\sin 2x = 2\sin x\cos x$, so $\sin x\cos x = \frac{1}{2}\sin 2x$. Substituting this into the Maclaurin series for $\sin x$ that we have seen over and over again gives us

$$\begin{aligned}\sin x\cos x &= \frac{1}{2}(2x - \frac{(2x^3)}{3!} + \frac{(2x)^5}{5!} - \cdots \\ &= \frac{1}{2}\sum_{i=0}^{n}(-1)^i 2^{2i+1}\frac{x^{2i+1}}{(2i+1)!} \\ &= \sum_{i=0}^{n}(-1)^i 2^{2i}\frac{x^{2i+1}}{(2i+1)!}.\end{aligned}$$

Again, this converges for all x because the power series for sin does.

(15) We could work this problem out by taking derivatives. It is too bad the problem did not ask us for $e^{-2(x+1)}$, though, because we could expand it in powers of $-2(x+1)$ by the usual power series for e^x, and we would have a power series centered around $x = -1$. For this reason we ask how are e^{-2x} and $e^{-2(x+1)}$ are related. Since $e^{-2(x+1)} = e^{-2x}e^{-2}$, we can write

$$e^{-2x} = e^2 e^{-2(x+1)} = e^2 \sum_{k=0}^{\infty}(-1^k)2^k \frac{(x+1)^k}{k!} = \sum_{k=0}^{\infty}(-1^k)e^2 \cdot 2^k \frac{(x+1)^k}{k!}.$$

(I've been changing the dummy summation index to remind you that using the letter n as the summation index is unimportant. You can use any letter you want; not using my symbol for a dummy variable is not a mistake!) The moral of the story is, when you see something that looks hard to do, but it isn't all that different from something easier to do, try to figure out how to change the hard thing into some expression involving the easier thing.