Directional Derivatives and the Gradient Vector

November 15, 2006

Directional Derivatives

- Recall:

$$
\begin{aligned}
f_x(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\
f_y(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.
\end{aligned}
$$

The Directional Derivative

- The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

  if this limit exists.

- If $\mathbf{u} = \mathbf{i} = \langle 1, 0\rangle$, then $D_{\mathbf{i}}f = f_x$, and if $\mathbf{u} = \mathbf{j} = \langle 0, 1\rangle$, then $D_{\mathbf{j}} = f_y$.

Theorem

- If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

- If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

Examples

- Find the directional derivative of

$$f(x, y) = x^3 - 3xy + 4y^2$$

at the point $(1, 2)$ in the direction $\theta = \pi/6$.

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- Find the directional derivative of $f(x, y) = xe^y + \cos(xy)$ at the point (2,0) in the direction of $\mathbf{v} = \mathbf{3i} - 4\mathbf{j}$.

The Gradient Vector

- If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

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- Example: find the gradient of $f(x,y) = \sin x + e^{xy}$ at $(0,1)$.

Fact:

- The equation of the directional derivative becomes:

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

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- **Example:** Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector $\mathbf{v} = \mathbf{2i} + \mathbf{5j}$.

Functions of three variables

- If $w = f(x, y, z)$ is a function of three variables, the **directional derivative** of f at (x_0, y_0, z_0) in the direction of the unit vector $\langle a, b, c\rangle$ is

$$D_u f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

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- Then

$$D_u f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c.$$

- The gradient is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- The formula for the directional derivative become

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

Example

Consider the function $f(x, y, z) = xy^2 + yz^3 + xy^2$.

- Find the gradient of f.

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- Find the gradient of f.
- Find the gradient of f at the point $(5, 4, -1)$.
- Find the rate of change of the function f at the point $(4, 5, -1)$ in the direction $\mathbf{u} = \langle 2/\sqrt{20}, -3/\sqrt{20}, -3/\sqrt{20} \rangle$.

Maximizing the Directional Derivative

- Suppose that f is a differentiable function of two (or three) variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(x,y)$ is $|\nabla f|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x)$.

Example

- If $f(x,y) = xe^y$, find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q(\frac{1}{2},2)$.

- In what direction does f have the maximum rate of change? What is this maximum rate of change?

Significance of the Gradient Vector

- The gradient ∇f gives the direction of fastest increase of f.
- The gradient Δf is orthogonal to the level surface S of f trough P.