

# Tangent Planes and Linear Approximations

November 10, 2006

# Tangent Planes

- Let  $S$  be a surface with equation  $z = f(x, y)$ .
- Let  $P(x_0, y_0, z_0)$  be a point on  $S$ .
- Let  $C_1$  and  $C_2$  be the curves obtained by intersecting the vertical planes  $y = y_0$  and  $x = x_0$  with the surface  $S$ .
- Let  $T_1$  and  $T_2$  be the tangent lines to the curves  $C_1$  and  $C_2$ .
- The **tangent plane** to the surface  $S$  at the point  $P$  is defined to be the plane that contains both tangent lines  $T_1$  and  $T_2$ .

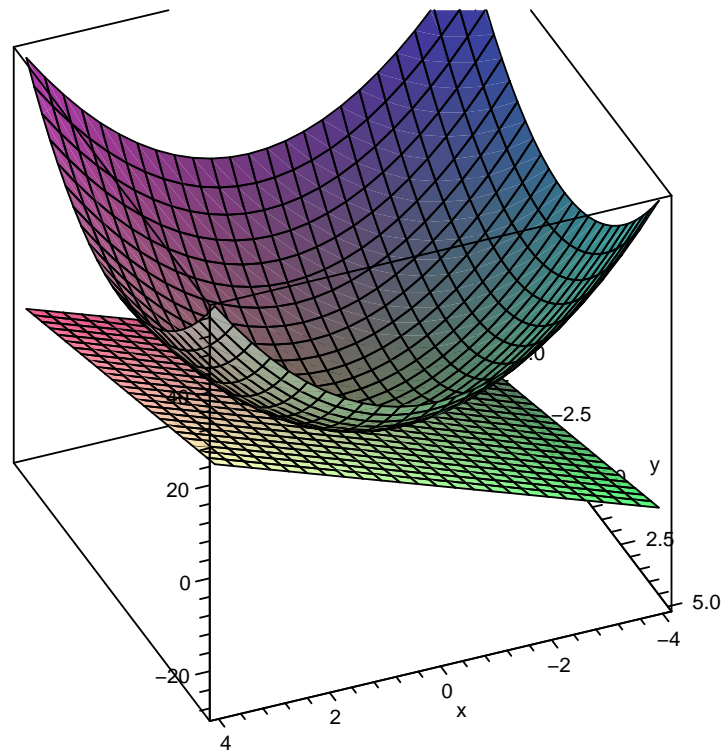
# Equations of the tangent plane

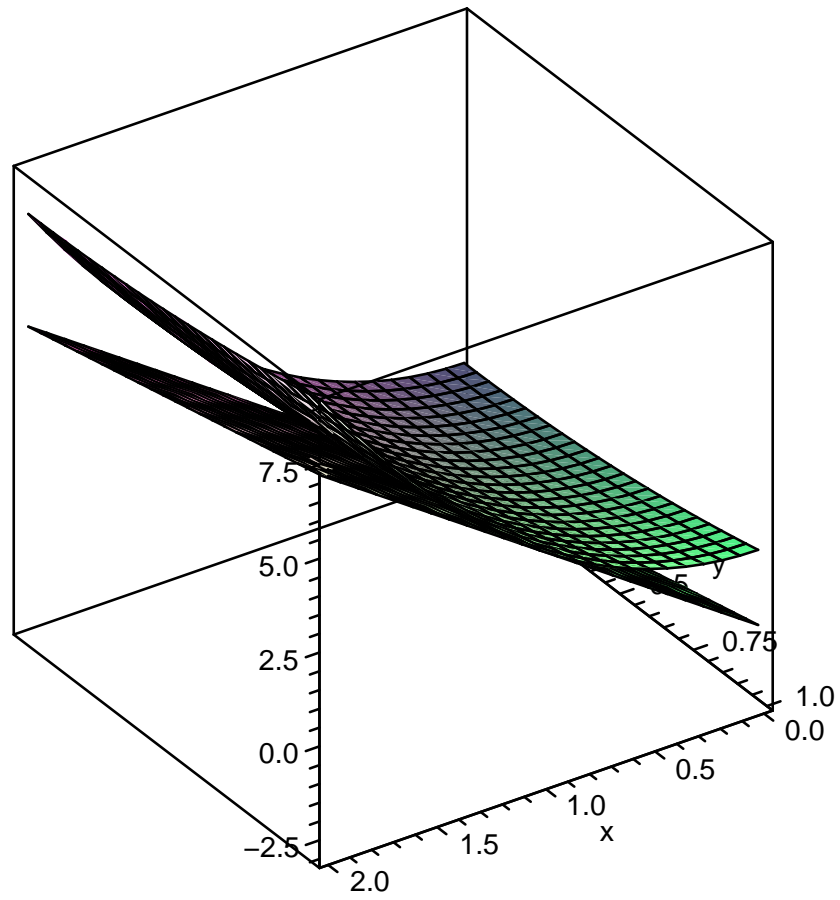
- Suppose  $f$  has a continuous partial derivatives.
- An equation of the tangent plane to the surface  $z = f(x, y)$  at the point  $P(x_0, y_0, z_0)$  is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

# Examples

- Find the tangent plane to the elliptic paraboloid  $z = 2x^2 + y^2$  at the point  $(1, 1, 3)$ .





# Linear Approximations

- The linear function whose graph is this tangent plane

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of  $f$  at  $(a, b)$  and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of  $f$  at  $(a, b)$ .

# Examples

- Find the linearization of the function  $f(x, y) = \sqrt{xy}$  at the point  $(4, 16)$ .

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- Find the linearization of the function  $f(x, y) = 1 + y + x \cos y$  at  $P_0(0, 0)$ .

## The increment of $z$

- Recall that for a function of one variable,  $y = f(x)$ , if  $x$  changes from  $a$  to  $a + \Delta x$ , we defined the increment of  $y$  as

$$\Delta y = f(a + \Delta x) - f(a).$$

- If  $f$  is differentiable at  $a$ , then

$$\Delta y = f'(a)\Delta x + \epsilon\Delta x,$$

where  $\epsilon \rightarrow 0$  as  $\Delta x \rightarrow 0$ .

- If  $z = f(x, y)$  and  $x$  changes from  $(a, b)$  to  $(a + \Delta x, b + \Delta y)$ , then the **increment** of  $z$  is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

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- If  $z = f(x, y)$ , then  $f$  is **differentiable** at  $(a, b)$  if  $\Delta z$  can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y,$$

where  $\epsilon_1$  and  $\epsilon_2 \rightarrow 0$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$ .

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- Example: Show that  $f(x, y) = xe^{xy}$  is differentiable at  $(1, 0)$  and find its linearization there.

# Differentials

- For a differentiable function  $z = f(x, y)$  we define the **differential**  $dz$ , also called the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy,$$

where the **differentials**  $dx$  and  $dy$  are independent variables.

- If  $dx = \Delta x = x - a$  and  $dy = \Delta y = y - b$  the the differential of  $z$  is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

# Examples

- If  $f(x, y) = x^2 + 3xy - y^2$ , find the differential  $dz$ .

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- If  $f(x, y) = x^2 + 3xy - y^2$ , find the differential  $dz$ .
- If  $x$  changes from 2 to 2.05 and  $y$  changes 3 to 2.96, compare the values of  $\Delta z$  and  $dz$ .