

Partial Derivatives

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Partial derivative of f with respect to x

- Let $f(x, y)$ be a function of two variables.
- Let $y = b$ be fixed.
- Then $g(x) = f(x, b)$ is a function of a single variable x .
- If g has a derivative at a , then we call it the **partial derivative of f with respect to x at (a, b)**

$$f_x(a, b) = g'(a)$$

Partial derivative of f with respect to y

- Now keep $x = a$ fix.
- Let $h(y) = f(a, y)$.
- If h has a derivative at b , then we call it the **partial derivative of f with respect to y at (a, b)**

$$f_y(a, b) = h'(b)$$

- By the definition of a derivative, we have

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h}$$

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b + h) - f(a, b)}{h}$$

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- The partial derivatives of $f(x, y)$ are the functions $f_x(x, y)$ and $f_y(x, y)$ obtained by letting the point (a, b) vary.

Notations

- If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of

$$z = f(x, y)$$

- To find f_x regard y as a constant and differentiate $f(x, y)$ with respect to x .
- To find f_y regard x as a constant and differentiate $f(x, y)$ with respect to y .

Examples

- If $f(x, y) = x^2 + 3x^3y - xy^2$ find $f_x(0, 1)$ and $f_y(1, 0)$

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$$f(x, y) = \frac{2y}{y + \cos x}$$

$$f(x, y) = e^{x^2+y^2+1}$$

$$f(x, y) = \ln(x + y)$$

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- Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Interpretations of Partial Derivatives

- Partial derivative can be interpreted as rates of change.
- The geometric interpretation: the partial derivatives are the slopes of the tangent lines at $P(a, b, c)$ to the curves given by the intersection of the surface given by $z = f(x, y)$ and the planes $x = a$ and $y = b$.

Higher Derivatives

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- If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables.
- So why stop here?
- The **second partial derivatives** of f are

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial^2 x} = \frac{\partial^2 z}{\partial^2 x}$$
$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \dots$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \dots$$

$$f_{yy} = \frac{\partial^2 f}{\partial^2 y}$$

- Example: Find the second derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Clairaut's Theorem

- Suppose f is defined on a disk D that contains the point (a, b) . If the functions f_{xy} and f_{yx} are both continuous on D , then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Partial derivatives of order 3 and higher

- Calculate f_{xxy} if $f(x, y) = \sin(3x^2 + xy)$.

More examples

- Find the partial derivatives of

$$f(x, y) = \int_x^y e^{t^2+t+1} dt$$

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- Find f_x, f_y, f_{xy}, f_{yx} for

$$f(x, y) = xye^{3xy}$$