Functions of Several Variables

November 6, 2006

Lecture 21

Functions of two variables

- A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$.

- The set D is the domain of f and its range is the set of values that f takes on.

- We also write $z = f(x, y)$

- The variables x and y are **independent variables** and z is the **dependent variable**.

Examples

- Find the domain of the function

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- Find the domain of the function

$$f(x, y) = \frac{2x + 3y}{x^2 + y^2 - 9}$$

- Find the domain and range of

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

Graphs

- If f is a function of two variables with domain , then the graph of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

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- **Example:** A **linear function** is a function

$$f(x) = ax + by + c$$

- The graph of such a function is a plane.

Examples

- $f(x, y) = \sin(x) + \sin(y)$

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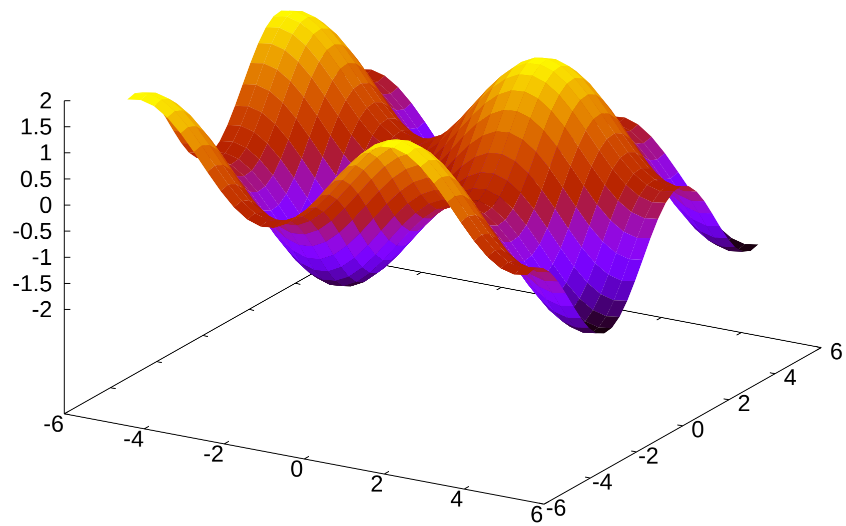
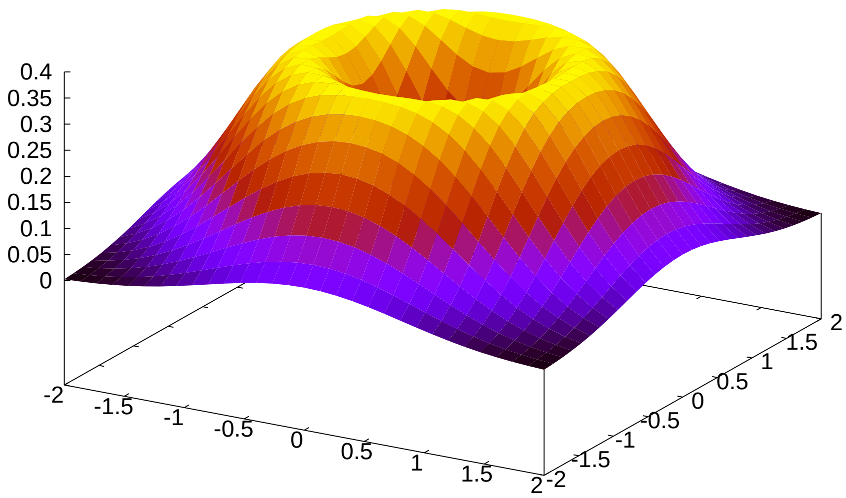


- $f(x,y) = (x^2 + y^2)e^{-x^2-y^2}$



The Cobb-Douglas production function

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The Cobb-Douglas production function

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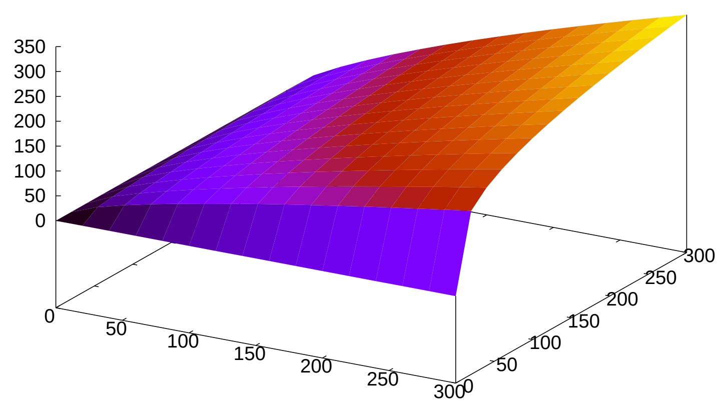
- $P(L, K) = 1.01L^{0.75}K^{0.25}$



Level Curves

- The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is constant.

Examples

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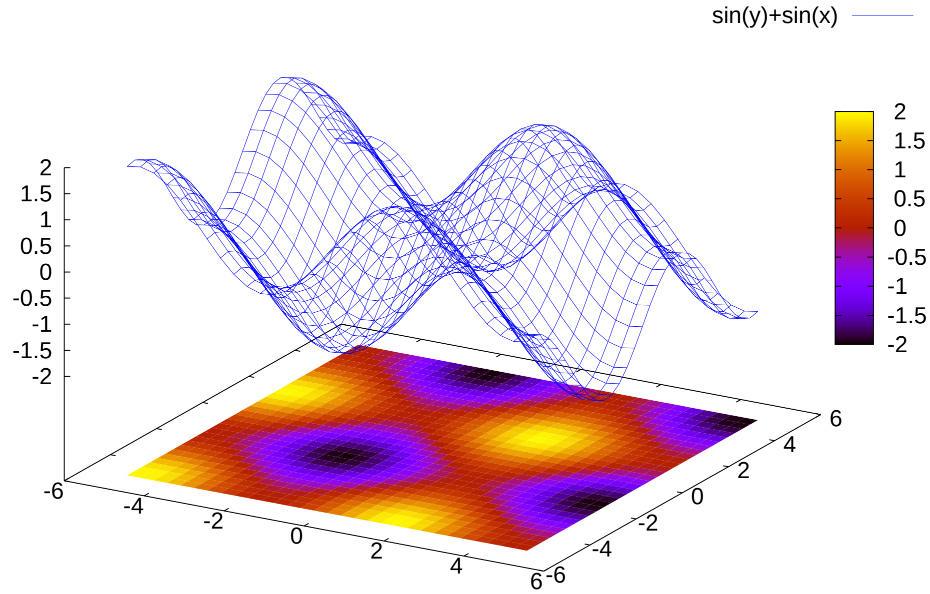

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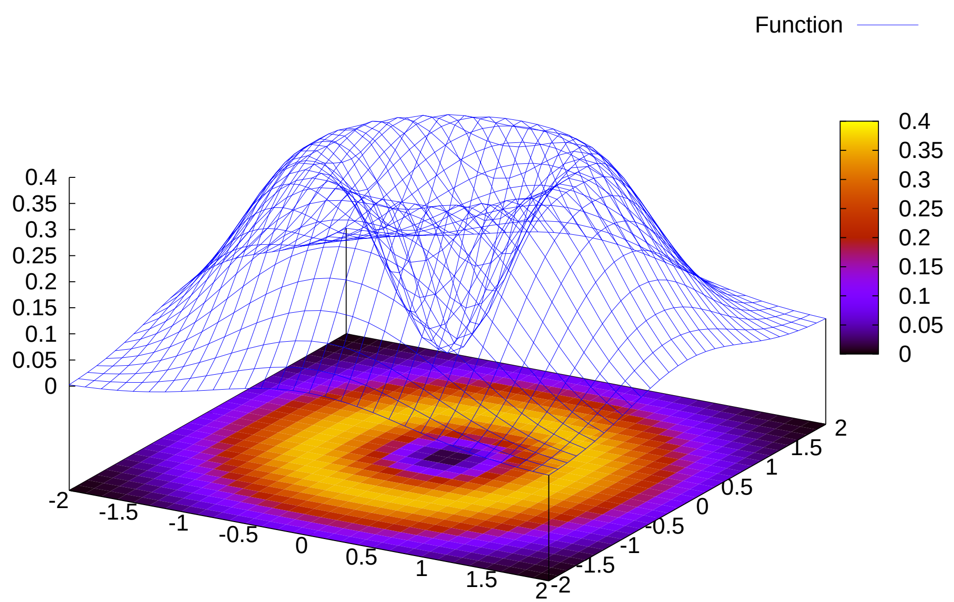

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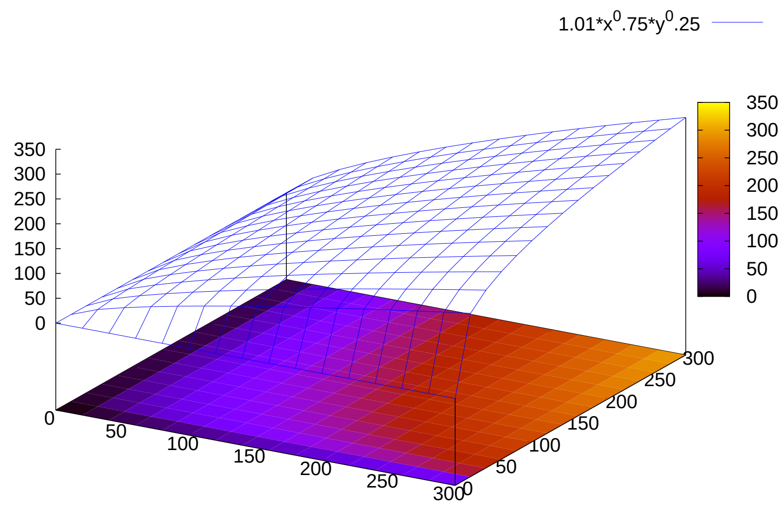

Limits and Continuity

- We say that a function $f(x, y)$ has limit L as (x, y) approaches a point (a, b) and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b).

- We write also $f(x, y) \to L$ as $(x, y) \to (a, b)$ and

$$\lim_{x\to a, y\to b} f(x, y) = L$$

- If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exists.

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- **Example:** Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.

Continuity

- A function f of two variables is called **continuous at** (a, b) if

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- **Examples:** polynomials, rational, trigonometric, exponential, logarithmic functions are continuous on theirs domain.

Example:

- Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 - 4y}{\sqrt{2x^2 - 4y + 1} - 1}$$

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- Find the largest set on which the function

$$\frac{2xy}{9 - x^2 - y^2}$$

is continuous.