

Vector Functions, Space Curves, Derivatives, and Integrals

November 1, 2006

Vector Functions

- A vector function is a function $\mathbf{r}(t)$ whose domain is the set of real numbers and whose range is a set of vectors – in general three-dimensional vectors.
- If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then they are called the **component functions** of \mathbf{r} .
- We write

$$\begin{aligned}\mathbf{r}(t) &= \langle f(t), g(t), h(t) \rangle \\ &= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}\end{aligned}$$

Limit of a vector function

- If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle$$

Example

- Find the limit

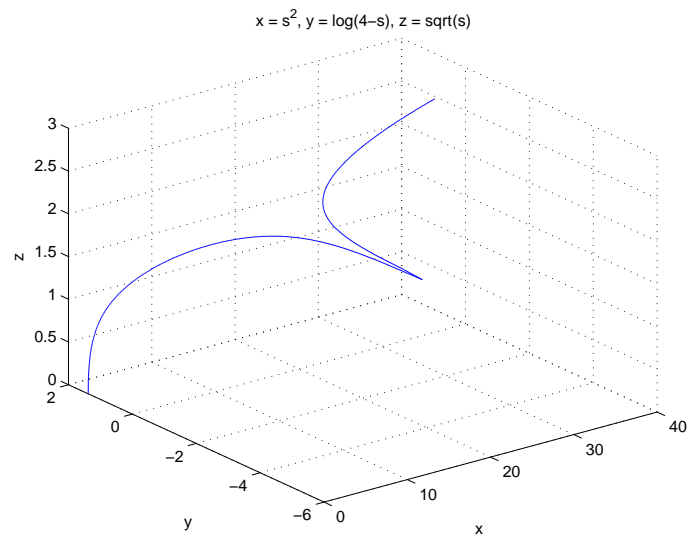
$$\lim_{t \rightarrow \pi/4} \langle \cos t, \sin t, t \rangle$$

Space Curves

- $\mathbf{r}(t) = \langle t^2, \ln(4 - t), \sqrt{t} \rangle$

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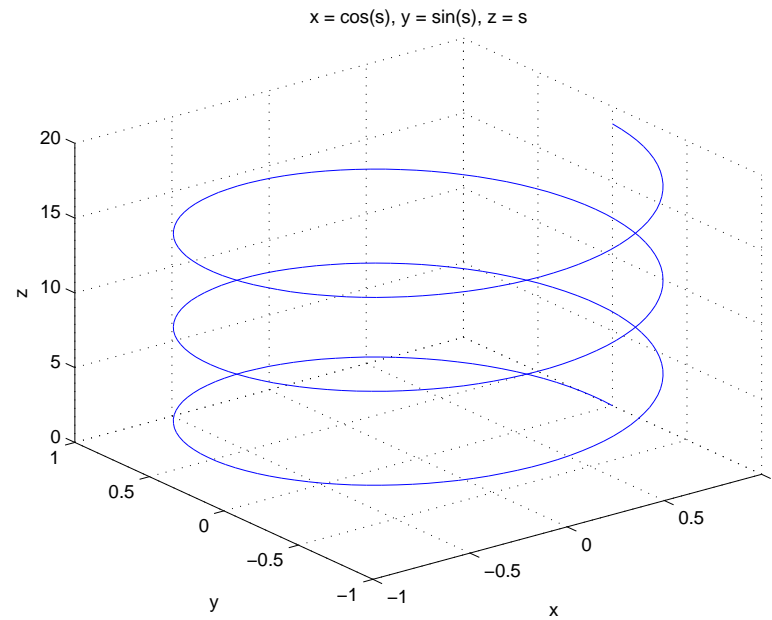


The Helix

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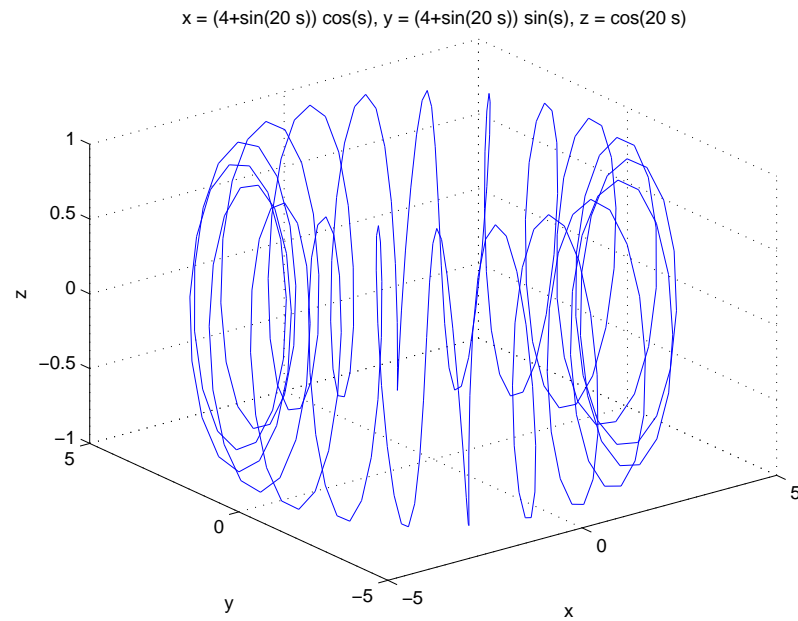


The Toroidal

- $\mathbf{r}(t) = (4 + \sin(2t)) \cos t \mathbf{i} + (4 + \sin 20t) \sin t \mathbf{j} + \cos 20t \mathbf{k}$

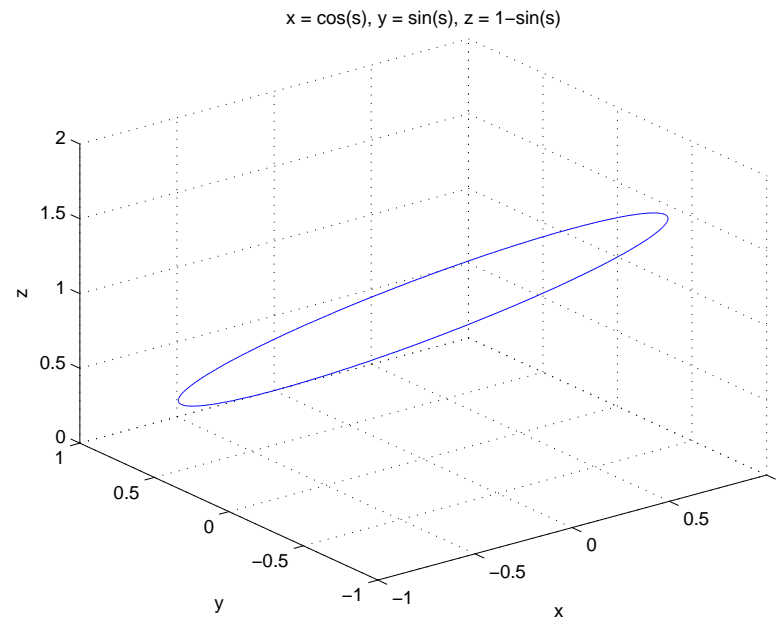
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Derivatives of Vector Functions

- The derivative \mathbf{r}' of \mathbf{r} is

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if the limit exists.

- The vector $\mathbf{r}'(t)$ is called the **tangent vector**.
- The **unit tangent vector** is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

How to compute $\mathbf{r}'(t)$?

- If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g , and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Examples

- Find the derivative of $\mathbf{r}(t) = (2t + t^2)\mathbf{i} + e^{-t^2}\mathbf{j} + \sin(2t)\mathbf{k}$.

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- Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2 \cos t \quad y = \sin t \quad z = t$$

at the point $(0, 1, \pi/2)$.

Differentiation Rules

$$[\mathbf{u}(t) + \mathbf{v}(t)]' = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$(c\mathbf{u}(t))' = c\mathbf{u}'(t)$$

$$(f(t)\mathbf{u}(t))' = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

$$(\mathbf{u}(t) \cdot \mathbf{v}(t))' = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

$$(\mathbf{u}(t) \times \mathbf{v}(t))' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

$$(\mathbf{u}(f(t)))' = f'(t)\mathbf{u}'(f(t)).$$

Integrals

- The definite integral

$$\int \mathbf{r}(t)dt = \left(\int_a^b f(t)dt \right) \mathbf{i} + \left(\int_a^b g(t)dt \right) \mathbf{j} + \left(\int_a^b h(t)dt \right) \mathbf{k}$$

- The Fundamental Theorem of Calculus

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t) \Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where \mathbf{R} is an antiderivative of \mathbf{r} .