The Dot Product and The Cross Product

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The Dot Product

- If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

- It is also called the **scalar product** or **inner product**.

Examples

- $\langle 2, 1 \rangle \cdot \langle -1, 3 \rangle$.

- $\langle 3, -2, 1 \rangle \cdot \langle 0, 1, 1 \rangle$.

Properties of the Dot Product

- If $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

  1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
  2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
  3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
  4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
  5. $\mathbf{0} \cdot \mathbf{a} = 0$.

The angle between two vectors

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- If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Orthogonal vectors

- $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Direction Angles

- The **direction angles** of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ that $\mathbf{a}$ makes with the positive $x-$,$y-$, and $z-$axes.

- The cosines of these direction angles are called the **direction cosines** of the the vector $\mathbf{a}$:

$$\cos\alpha = \frac{a_1}{|\mathbf{a}|},\ \cos\beta = \frac{a_2}{|\mathbf{a}|},\ \cos\gamma = \frac{a_3}{|\mathbf{a}|}.$$

Projections

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- Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$:

$$\mathrm{comp}_a \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

- Vector projection of $\mathbf{b}$ onto $\mathbf{a}$

$$\mathrm{proj}_a b = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$$

Work done by a constant force

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- Example: A constant force $\mathbf{F} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ moves an object along a straight line from the point $(1, 0, 0)$ to $(-3, 2, 3)$. Find the work done.

The Cross Product

The Cross Product

- If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

- The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

- If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$|a \times b| = |a||b| \sin \theta$$

- Two nonzero vectors are parallel if and only if $\mathbf{a} \times \mathbf{b} = 0$

- The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Examples

- Find a vector perpendicular to both $\langle -2, 2, 0 \rangle$ and $\langle 0, 1, 2 \rangle$ of the form $\langle 1, \underline{\hspace{2em}}, \underline{\hspace{2em}} \rangle$

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- Find the area of the triangle with vertices $P(0, 0, 0)$, $Q(-2, 2, 5)$, $R(0, 3, -3)$.

The volume of a parallelepiped

- The volume of a parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product:

$$V = |a \cdot (b \times c)|$$

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- Find the volume of the parallelepiped with adjacent edges PQ, PR, PS where$P(1, 4, -3)$, $Q(3, 7, 0)$, $R(0, 3, -4)$, $S(7, 2, -1)$.