

# Taylor and Maclaurin Series

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# Recall

- The **Taylor series** of the function  $f$  at  $a$  is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

- The **Maclaurin series**

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

## When does a function equals its Taylor series?

- Consider the partial sums of the Taylor series:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n.$$

- $T_n$  is called the  **$n$ th-degree Taylor polynomial of  $f$  at  $a$** .
- Then  $f$  equals its Taylor series if  $f(x) = \lim_{n \rightarrow \infty} T_n(x)$ .

# The remainder of the Taylor series

- Let  $R_n(x) = f(x) - T_n(x)$

- If

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

for  $|x - a| < R$ , then  $f$  is equal to the sum of its Taylor series on the interval  $|x - a| < R$ .

## Useful formulas

- **(Taylor's inequality)** If  $|f^{(n+1)}(x)| \leq M$  for  $|x - a| \leq d$ , then the remainder  $R_n(x)$  satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1}$$

for  $|x - a| \leq d$ .

- Often we use the following fact:

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$$

for every real number  $x$ .

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- Find the Maclaurin series for  $\sin x$  and prove that it represents  $\sin x$  for all  $x$ .

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

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- Find the Maclaurin series for  $x^2 \cos x$ .
- Evaluate  $\int e^{-x^2} dx$  as an infinite series.

- Use the Maclaurin series for  $e^x$  to evaluate

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- Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{5^n}{3^n n!}$$