Power Series

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A note about series

- Recall that a series is an "infinite sum"

$$\sum_{n=1}^{\infty} a_n.$$

- A series is **absolutely convergent (AC)** if the series

$$\sum_{n=1}^{\infty} |a_n|$$

converges.

- An absolutely convergent series is also convergent, in the sense that $\sum a_n$ converges as well.

- Examples of AC series are

$$\sum \frac{(-1)^n}{n^2}, \sum \frac{1}{n^2}, \sum \frac{1}{n!}, \ldots$$

- A series is **conditionally convergent (CC)** if it is convergent but not absolutely convergent.

- Note that that a series can be convergent and fail to be AC (that is, CC) only if it contains negative terms as well; the most common examples are the alternating series.

- An example of a CC series

$$\sum \frac{(-1)^n}{n}.$$

Test for convergence

- The idea is that you first want to test for AC using either the Ratio test or the Root test. Both these tests are for AC or divergence (D).

- If both these tests are inconclusive (the corresponding limit equals 1), than you should apply the comparison test, or the integral test to the series
$$\sum |a_n|.$$

- If this series is convergent, then the original series is AC.

- If this series is divergent, but the original series is convergent (using the Alternating series test, for example), then the series is CC.

Power Series

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- Example: if $c_n = 1$ for all $n \in \mathbb{N}$, then the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n,$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Power series about a

- A series of the form

$$\sum_{n=0}^{n} c_n(x-a)^n = c_0 + c_1(x-1) + c_2(x-a)^2 + \cdots$$

  is called **a power series in** $(x-a)$ or a **power series centered at a** or a **power series about** a.

- The question is: For which values of x is a power series convergent and for which is divergent?

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- $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}} (x+3)^n$

Radius of convergence

For a given power series $\sum_{n=0}^{\infty} c_n(x-1)^n$ there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The number R is called the radius of convergence ($R = 0$ in the first case and $R = \infty$ in the second case).

Important

If x is an endpoint

$$x = a \pm R$$

anything can happen: the series might converge at one or both endpoints or it might diverge at both endpoints.

More examples

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- $\sum_{n=1}^{\infty} n^n x^n$
- $\sum_{n=1}^{\infty} \frac{n(x-4)^n}{n^3+1}$