

①-④ Find the radius of convergence and interval of convergence of the series.

① $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ $a_n = \frac{(-1)^n x^n}{n+1}$ We use the Ratio Test to

see when the series converges. $\lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{n+2}}{\frac{x^n}{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right| =$

$\lim_{n \rightarrow \infty} \frac{|x|}{1 + \frac{1}{n+1}} = |x|$. So by the Ratio Test, we know the series

converges when $|x| < 1$, so $R=1$. Now to find I . When $x=-1$, the series is the harmonic series, and so it diverges. When $x=1$, the series is the Alternating Harmonic series, and so it converges by the Alternating Series test. So we have $I = (-1, 1]$.

② $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ $a_n = \frac{x^n}{n3^n}$ We shall follow the same method as above.

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{x^n}}{\frac{x^n}{n3^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x n}{(n+1)3} \right| = \frac{|x|}{3} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{|x|}{3}$

So the series converges when $\frac{|x|}{3} < 1$, $|x| < 3$. So $R=3$.

When $x=-3$, the series is the alternating harmonic series, which we know converges by the Alternating Series Test. When $x=3$, the series is the harmonic series which we know diverges. So $I = [-3, 3)$

③ $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ We proceed as before.

$$a_n = \frac{(-1)^n x^{2n}}{(2n)!} \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = \lim_{n \rightarrow \infty} \frac{|x|^2}{(2n+1)(2n+2)} = 0$$

So, by the Ratio test, this series converges for all real numbers.
So $R = \infty$ & $I = (-\infty, \infty)$.

④ $\sum_{n=0}^{\infty} n^3 (x-5)^n$ We proceed as before.

$$a_n = n^3 (x-5)^n \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3 (x-5)^{n+1}}{n^3 (x-5)^n} \right| = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^3 |x-5|$$

$= |x-5|$. By the Ratio Test, the series converges when $|x-5| < 1 \Leftrightarrow -1 < x-5 < 1 \Leftrightarrow 4 < x < 6$ So $R = 1$.

When $x=4$, the series is $\sum_{n=0}^{\infty} (-1)^n n^3$ which diverges.

When $x=6$, the series is $\sum_{n=0}^{\infty} n^3$ which also diverges. So $I = (4, 6)$

⑤ & ⑥: Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$.

What can be said about the convergence or divergence of the following series?

⑤ $\sum_{n=0}^{\infty} c_n 8^n$ Here $x=8$. We know that the series above diverges at least when $x < -6$ or $x \geq 6$. Since $8 > 6$, this series is divergent.

⑥ $\sum_{n=0}^{\infty} c_n (-3)^n$ Here $x=-3$. We know the above series converges at least when $-4 < x < 4$. Since $-3 \geq -4$ & $-3 < 4$, this series is convergent.

⑦ & ⑧

Find a power series representation for the function & determine the interval of convergence

$$\textcircled{7} \quad f(x) = \frac{3}{1-x^4} = 3 \left(\frac{1}{1-x^4} \right) = 3(1+x^4+x^8+x^{12}+x^{16}+\dots) = 3 \sum_{n=0}^{\infty} (x^4)^n$$

$$= \sum_{n=0}^{\infty} 3x^{4n} \quad \text{This converges} \Leftrightarrow \sum_{n=0}^{\infty} (x^4)^n \text{ converges. So } \sum_{n=0}^{\infty} 3x^{4n}$$

converges when $|x^4| < 1 \Leftrightarrow |x| < 1$, so $R=1$ & $I=(-1, 1)$

$$\textcircled{8} \quad f(x) = \frac{x}{4x+1} = x \left(\frac{1}{1-(-4x)} \right) = x \sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{n+1}$$

This converges when $| -4x | < 1 \Leftrightarrow |x| < \frac{1}{4}$ $R = \frac{1}{4}$ $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$

⑨

Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2} \quad \text{What is the radius of convergence?}$$

$$\text{Note that } \frac{d}{dx} \left(\frac{-1}{1+x} \right) = \frac{1}{(1+x)^2}$$

$$\frac{-1}{1+x} = \frac{-1}{1-(-x)} = - \sum_{n=0}^{\infty} (-x)^n = - \sum_{n=0}^{\infty} (-1)^n (x)^n$$

$$\text{So } f(x) = \frac{1}{(1+x)^2} = \frac{-d}{dx} \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} = \sum_{n=0}^{\infty} (-1)^{n+2} (n+1) x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \quad \text{Here, } R=1, \text{ since } | -x | < 1 \Leftrightarrow |x| < 1.$$

(8) Use part (a) to find a power series for $f(x) = \frac{1}{(1+x)^3}$

$$\text{Note that } \frac{d}{dx} \frac{1}{(1+x)^2} = -2 \left(\frac{1}{(1+x)^3} \right)$$

$$\text{So } f(x) = \frac{1}{(1+x)^3} = \frac{-1}{2} \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} (n+1) n x^{n-1}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n \quad \text{Again, } R=1$$

(9) Use part (b) to find a power series for $f(x) = \frac{x^2}{(1+x)^3}$

$$f(x) = x^2 \cdot \frac{1}{(1+x)^3} = x^2 \cdot \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^{n+2}$$

We can rewrite this as $\frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n)(n-1) x^n$. Note the change

in the initial value of n .

(10) Find a power series representation for $f(x) = \ln(1+x)$. What is the radius of convergence

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n \quad (\text{Here } R=1)$$

$$\text{So, } f(x) = \ln(1+x) = \int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty} (-1)^n x^n dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} + C \quad f(0) = \ln(1) = 0, \text{ so } C=0. \quad \& R=1.$$

⑪ Use ⑩ to find a power series for $f(x) = x \ln(1+x)$.

$$f(x) = x \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n} = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n-1} \quad (R=1)$$