

Math 81. *Abstract Algebra*.

Homework 6. Due on Wednesday, 2/25/2009.

1. Pages 581-2, problems 3, 4, 9.
2. Let K/F be a finite separable extension.
 - (a) Show that there is a “smallest” finite extension L of K with L/F Galois. L is called the Galois closure of K/F .
 - (b) Determine the Galois closure L of $\mathbb{Q}(\sqrt[3]{2}, \sqrt[5]{2})/\mathbb{Q}$, and compute its degree over \mathbb{Q} .
 - (c) For L as in the previous part, determine whether or not $\text{Gal}(L/\mathbb{Q})$ is abelian.
3. Suppose that K/F is a finite Galois extension of degree n with Galois group $G = \{\sigma_1, \dots, \sigma_n\}$. For an element $\alpha \in K$, define its trace and norm as follows:

$$\text{Tr}(\alpha) = \text{Tr}_{K/F}(\alpha) = \sigma_1(\alpha) + \dots + \sigma_n(\alpha), \quad N(\alpha) = N_{K/F}(\alpha) = \sigma_1(\alpha) \dots \sigma_n(\alpha).$$

- (a) Show that Tr and N map K to F , and satisfy $\text{Tr}(\alpha + \beta) = \text{Tr}(\alpha) + \text{Tr}(\beta)$ and $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in K$.
- (b) Show that Tr is a surjective map. [*Hint: first show that there is an element $\alpha \in K$ for which $\text{Tr}(\alpha)$ is not zero. Note that in characteristic 0 or characteristic p with p not dividing n , this is easy, but there is a general way to do this in all cases.*]