

Dartmouth College

Mathematics 81

This problem is part of the assignment due on Wednesday, 15 January.

1. Let $\mathbb{Z}_{(p)}$ denote the ring \mathbb{Z} localized at the prime ideal $p\mathbb{Z}$, that is, if $R = \mathbb{Z}$ and $S = \mathbb{Z} \setminus p\mathbb{Z}$, then $\mathbb{Z}_{(p)} = S^{-1}R$.

- (a) Characterize $\mathbb{Z}_{(p)}$ as a subset of \mathbb{Q} , that is

$$\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid \text{put your conditions here}\}$$

- (b) Characterize the unit group $\mathbb{Z}_{(p)}^\times$.

- (c) Show that every nonzero element of $\mathbb{Z}_{(p)}$ can be written as $p^\nu u$, where ν is a nonnegative integer, and $u \in \mathbb{Z}_{(p)}^\times$.

- (d) Characterize all the ideals of $\mathbb{Z}_{(p)}$ (Hint: Show $\mathbb{Z}_{(p)}$ is a PID). Conclude that $\mathbb{Z}_{(p)}$ has a unique maximal ideal, which makes it an example of a *local ring*.

- (e) Show that $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$.