Dartmouth College

Mathematics 81

This problem is due on Wednesday, 10 January.

1. Let $\mathbb{Z}_{(p)}$ denote the ring $\mathbb{Z}$ localized at the prime ideal $p\mathbb{Z}$, that is, if $R = \mathbb{Z}$ and $S = \mathbb{Z} \setminus p\mathbb{Z}$, then $\mathbb{Z}_{(p)} = S^{-1}R$.

   (a) Characterize $\mathbb{Z}_{(p)}$ as a subset of $\mathbb{Q}$, that is

   $$\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid \text{your conditions here}\}$$

   (b) Characterize the unit group $\mathbb{Z}_{(p)}^{\times}$.

   (c) Show that every nonzero element of $\mathbb{Z}_{(p)}$ can be written as $p^{\nu}u$, where ν is a nonnegative integer, and $u \in \mathbb{Z}_{(p)}^{\times}$.

   (d) Characterize all the ideals of $\mathbb{Z}_{(p)}$ (hint: Show $\mathbb{Z}_{(p)}$ is a PID). Conclude that $\mathbb{Z}_{(p)}$ has a unique maximal ideal, which makes it an example of a *local ring*.

   (e) Show that $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$.