Dartmouth College

Mathematics 81

Homework assigned Wednesday, January 17

1. Let $\zeta = e^{2\pi i/8}$ be a primitive eighth root of unity.

   (a) Show that $(\zeta + \zeta^{-1})^2 = 2$.

   (b) Show that $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\zeta)$.

   (c) Compute the degree $[\mathbb{Q}(\zeta) : \mathbb{Q}(\sqrt{2})]$.

2. Show that $x^3 - 2$ is irreducible over $\mathbb{Q}(i)$, $i = \sqrt{-1} \in \mathbb{C}$. Do not attempt to factor the polynomial; argue via field extensions.

3. Let $m_1, m_2, \ldots, m_t$ be integers.

   (a) Show that $[\mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_t}) : \mathbb{Q}] \le 2^t$.

   (b) Give an example to show the inequality can be strict, and justify by computing degrees.

   (c) Now assume the the integers m_i are square-free and are coprime in pairs. Show that $[\mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_t}) : \mathbb{Q}] = 2^t$. Hint: Induction on t. You proably want to work out the case $t = 2$ carefully before trying the general argument.