

- Differentiate to solve the following for the function $u(t)$.

[Assume f' exists
and $a \neq 0$]

$$\int_0^t y u(y) dy - a u(t) = f(t) \quad \text{on } 0 \leq t \leq 1$$

[Hint: convert to ODE; what is the IC?]

- Prove the lemma.
$$\int_a^x \int_a^s f(y) dy ds = \int_a^x f(y) (x-y) dy$$

[Hint: define $F(s) := \int_a^s f(y) dy$

& write $\int_a^x F(s) ds$ as $\int_a^x \underbrace{1}_u \cdot \underbrace{F(s)}_{v'} ds$]

MATH 46 WORKSHEET: Volterra integral equations

Barnett
4/30/08

SOLUTIONS

Differentiate to solve the following for the function $u(t)$.

[Assume f' exists and $a \neq 0$]

This was #11 on p. 245

$$\int_0^t y u(y) dy - a u(t) = f(t) \quad \text{on } 0 \leq t \leq 1$$

[Hint: convert to ODE; what is the IC?]

$$\left\{ \frac{d}{dt} \right.$$

$$t u(t) - a u'(t) = f'(t)$$

put in std 1st order lin form:

$$u' - \frac{t}{a} u = -\frac{f'(t)}{a} \quad \text{ODE}$$

IC get by sub $t=0$ into integral eqn:

$$\int_0^0 y u(y) dy - a u(0) = f(0) \Rightarrow u(0) = -\frac{f(0)}{a}$$

Solve IVP:

integrating factor $p(t) = e^{\int p(t) dt} = e^{-\frac{t^2}{2a}}$

$$u = \frac{1}{p} \left[\int Sng + c \right] = e^{\frac{t^2}{2a}} \left[\int e^{-\frac{t^2}{2a}} \frac{f'(t)}{-a} dt + c \right]$$

$$= -a e^{\frac{t^2}{2a}} \int_0^t e^{-s^2/2a} f'(s) ds + c e^{\frac{t^2}{2a}}$$

match ICs gives

$$u(t) = -a e^{\frac{t^2}{2a}} \left[\int_0^t e^{-s^2/2a} f'(s) ds + f(0) \right]$$

Prove the lemma.

$$\int_a^x \int_a^s f(y) dy ds = \int_a^x f(y) (x-y) dy$$

[Hint: define $F(s) := \int_a^s f(y) dy$

& write $\int_a^x F(s) ds$ as $\int_a^x \underbrace{1}_{u'} \cdot \underbrace{F(s)}_{v} ds$

note $v' = f$ which is useful.

sorry! →

$$\int_a^x \underbrace{1}_{u'} \cdot \underbrace{F(s)}_{v} ds = - \int_a^x \underbrace{u}_{s} \underbrace{v'}_{f(s)} ds + \underbrace{uv}_{[sF(s)]}_a^x$$

$$= - \int_a^x s f(s) ds + x \int_a^x f(s) ds$$

$$= \int_a^x f(s) (x-s) ds$$

or could replace s with y .