MATH 46 WORKSHEET: Dimensional Analysis II

Th 3/27/08

~ SOLUTIONS ~

Here we explore the fundamental soln for the heat equation – without calculus!

Pulse of energy sized e released at origin at time $t=0$. The medium has heat capacity c (energy per volume per degree) and thermal conductivity $\mathcal{K}$ (power per length per degree). The temperature at distance r and time t is u (we take $u=0$ everywhere for $t<0$)

$t=0$ $t=1$ $t=2$ $t=3$

$$\begin{array}{c|c} & \begin{matrix} e & r & t & u & c & \mathcal{K} \end{matrix} \\ \begin{matrix} E \\ L \\ T \\ \Theta \end{matrix} & \begin{bmatrix} 1 & & & & 1 & 1 \\ & 1 & & & -3 & -1 \\ & & 1 & & & -1 \\ & & & 1 & -1 & -1 \end{bmatrix} \end{array}$$

a) Using fundamental units energy (E), Length (L), time (T), temperature (Θ), fill in the dimensions of the $m=6$ quantities in the problem above (check with me).

b) Find $p=2$ independent dimensionless quantities. Since there's freedom, choose

Π_1 to not involve u: $\Pi_1 = \cdots \dfrac{cr^2}{\mathcal{K}t}$ ... note it's $\dfrac{\text{dist}^2}{\text{time}}$

Π_2 to not involve r: $\Pi_2 = \cdots \dfrac{ce^2}{\mathcal{K}^3 t^3 u^2}$ or $\dfrac{(\mathcal{K}t)^{3/2} u}{c^{1/2} e}$ if choose power of u to be 1

c) Buckingham Pi Theorem tells us $F(\Pi_1, \Pi_2) = 0$, so $\Pi_2 = g(\Pi_1)$

From this derive a solution of the form $u = \left(\dfrac{ce^{1/2}}{(\mathcal{K}t)^{3/2}}\right) \tilde{g}\left(\dfrac{cr^2}{\mathcal{K}t}\right)$

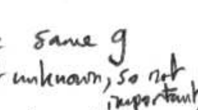
just be rearranging $\Pi_2 = g(\Pi_1)$

note not the same g as before (but unknown, so not important)

d) if $r=0$ how must u scale with t? (you may assume g has limits at $0, \infty$; in fact PDEs tell us g is a gaussian!)

$\tilde{g}(0)$ or $\tilde{g}(\infty)$ is some const.

$\Rightarrow u = (\text{const}) \dfrac{ce^{1/2}}{(\mathcal{K}t)^{3/2}} \sim t^{-3/2}$ if all else held const.



e) How does scaling in d) change in general space dimension d? (we had $d=3$ above; note that $\mathcal{K}$ has units $ET^{-1}L^{2-d}\Theta^{-1}$ in general d) Turns out $u \sim t^{-d/2}$ but have to redo lin. algebra.

Note $g(\Pi_1) = e^{-\Pi_1^2/2}$ a result we'll get to later.

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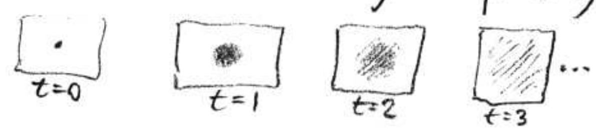

$$
\begin{array}{c|cccccc}
 & e & r & t & u & c & \mathcal{K} \\
\hline
E & 1 & & & & 1 & 1 \\
L & & 1 & & & -3 & -1 \\
T & & & 1 & & & -1 \\
\Theta & & & & 1 & -1 & -1
\end{array}
$$

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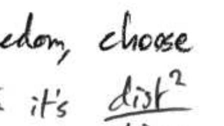

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