

MATH 46 WORKSHEET : Image (de)convolution in 1d

9/7/08
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Consider symm blurring operator $Kf(x) = \int_{-\pi}^{\pi} k(x-y) f(y) dy$, $k(s) =$ even symm \uparrow 2π -periodic 'aperture func.'

A) Show that $\phi_0(x) = 1$ is an eigenfunction of K , and find its eigenvalue λ_0 .
[Hint: why is $K\phi_0(x)$ indep. of x ? why is λ_0 indep. of x ?]

B) Show that $\phi_n(x) = \cos nx$, $n=1,2,\dots$ is eigenfunc. of K , find its eigenvalue λ_n .
[Hint: addition formula, even]

C) How do λ_n relate to Fourier cos coeffs k_n of aperture func $k(s)$?

You could check that $\sin nx$ is also eigenfunc. w/ same eigenval. λ_n .

Assume image is $f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$

& after blurring $Kf(x) = g(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx]$

D) How are g 's Fourier coeffs related to those of f ?

Such is the nature of convolution kernels. How would you invert $g \rightarrow f$, i.e. deconvolve?

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SOLUTIONS

Consider symm blurring operator $Kf(x) = \int_{-\pi}^{\pi} k(x-y) f(y) dy$, $k(s) =$ even symm \uparrow 2π -periodic 'aperture func.'

A) Show that $\phi_0(x) = 1$ is an eigenfunction of K , and find its eigenvalue λ_0 .
[Hint: why is $K\phi_0(x)$ indep. of x ? why is λ_0 indep. of x ?]

$$(K1)(x) = \int_{-\pi}^{\pi} k(x-y) \cdot 1 \cdot dy \xrightarrow[\text{change var.}]{s=y-x} \int_{-\pi-x}^{\pi-x} k(-s) ds \xrightarrow[\text{periodicity \& even-symm}]{=} \int_{-\pi}^{\pi} k(s) ds \quad \text{obviously const w.r.t. } x$$

so $\lambda_0 =$ this const $= \int_{-\pi}^{\pi} k(s) ds$

B) Show that $\phi_n(x) = \cos nx$, $n=1,2,\dots$ is eigenfunc. of K , find its eigenvalue λ_n .
[Hint: addition formula, kernel]

$$(K\phi_n)(x) = \int_{-\pi}^{\pi} k(x-y) \cos ny \, dy = \int_{-\pi-x}^{+\pi-x} k(-s) \cos n(s+x) \, ds \xrightarrow[\text{bring out}]{=} \int_{-\pi}^{\pi} k(s) \cos ns \cos nx \, ds - \int_{-\pi}^{\pi} k(s) \sin ns \sin nx \, ds = \frac{\phi_n(x)}{\cos nx} \cdot \int_{-\pi}^{\pi} k(s) \cos ns \, ds \xrightarrow[\text{bring out}]{=} \lambda_n \phi_n(x)$$

periodic so can shift. limit to $(-\pi, \pi]$.

zero since k even symm

C) How do λ_n relate to Fourier cos coeffs k_n of aperture func $k(s)$? $\lambda_n = \pi k_n$

You could check that $\sin nx$ is also eigenfunc. w/ same eigenval. λ_n . Since $k_n = \frac{1}{\pi} \int_{-\pi}^{\pi} k(s) \cos ns \, ds$ Euler-Fourier or projection formula.

Assume image is $f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$

& after blurring $Kf(x) = g(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx]$

D) How are g 's Fourier coeffs related to those of f ?

Fourier basis = eigenbasis for K , so action of K is multiplication in this basis:

$$\left. \begin{aligned} A_0 &= \lambda_0 a_0 = \pi k_0 a_0 \\ \text{for } n=1,2,\dots \left\{ \begin{aligned} A_n &= \lambda_n a_n = \pi k_n a_n \\ B_n &= \lambda_n b_n = \pi k_n b_n \end{aligned} \right. \end{aligned} \right\} \text{ so Fourier coeffs get multiplied by } \pi \times \text{ Fourier coeffs of aperture func.}$$

Such is the nature of convolution kernels. How would you invert $g \rightarrow f$, i.e. deconvolve? divide Fourier coeffs by πk_n .