

Integral Equations

Volterra operator $Ku(x) = \int_0^x k(x,y) u(y) dy$

'lower-triangular' kernel


- $u - \lambda Ku = f$ has unique solution for any λ , continuous func. f .
This is $u = (1 + \lambda K + \lambda^2 K^2 + \dots) f$. Neumann Series (converges)
- this tells you K has no eigenvalues
- The other way to solve a Volterra eqn. is to take derivs. via Leibniz's formula until you get an ODE, then solve that, with ICs that can be extracted from the $x \rightarrow 0$ limit of the integral eqn.
- You can go backwards, i.e. given an ODE convert to Volterra eqn. Make sure you can do this for 1st order. (Don't do 2nd order since harder, see p.283)
- Volterra eqns arise in real-world situations where $u(t)$ determined by history $u(s)$ for $s < t$.

Fredholm degenerate

op. $Ku(x) = \sum_{j=1}^N \alpha_j(x) \beta_j(y)$ $\{\alpha_j\}$ L.I. set
 $\{\beta_j\}$ " "
 $\Rightarrow K$ need not be symm.

- Eigenvalues are those of matrix A with entries $A_{ij} = (\beta_i, \alpha_j)$, plus an ∞ -multiplicity zero eigenvalue.
- Eigenfuncs are $\sum_{j=1}^N c_j \alpha_j(x)$ where \vec{c} is corresponding eigenvector of A , plus the set of all funcs orthogonal to all $\{\beta_j\}$ forms the zero eigenspace.
- $Ku - \lambda u = f$ has unique soln if $\lambda \neq$ eigenvalue, which can be got from $\sum_{j=1}^N \alpha_j(x) c_j - \lambda u(x) = f(x)$ (*)
But if $\lambda = j^{\text{th}}$ eigenvalue then no soln. unless $\vec{f} = \{f_i\}$ $f_i = (\beta_i, f)$ is in the range of $A\vec{c} - \lambda\vec{c}$, i.e. $A\vec{c} - \lambda\vec{c} = \vec{f}$ consistent.
- $Ku = f$ has no soln. unless f is in $\text{Span}\{\alpha_j\}$ \leftarrow the range of K . when soln is nonunique in the zero eigenspace component.

Fredholm symmetric

op. $Ku(x) = \int_a^b k(x,y) u(y) dy$

with $k(y,x) = k(x,y)$ continuous.

so $(Ku, v) = (u, Kv) \quad \forall u, v \in L^2$.

- Eigenvalues λ_j real, tend to zero, a number of them.
- Eigenfunctions ϕ_j orthogonal, complete in L^2 ... means form a basis for L^2 .

All your techniques from symmetric matrices work:

• $Ku - \lambda u = f$ write $u = \sum_{j=1}^{\infty} c_j \phi_j$, $f = \sum_{j=1}^{\infty} f_j \phi_j$

gives $c_j = \frac{f_j}{\lambda_j - \lambda}$ by orthogonality \uparrow you may need to compute.

\hookrightarrow tells you: unique soln. if $\lambda \neq$ eigenvalue

otherwise nonunique solution: $u(x) = c \phi_j(x) + \sum_{i \neq j} \frac{f_i \phi_i(x)}{\lambda_i - \lambda}$

(if $\lambda = \lambda_j$ for some j) \uparrow arbitrary. if $f_i = 0$.

or no solution if $f_i \neq 0$.

• This all applies for $\lambda = 0$ too.

• K operator can be written in spectral form $K = \sum_{j=1}^{\infty} \lambda_j \underbrace{\phi_j(\phi_j, \cdot)}_{\text{projection onto } j^{\text{th}} \text{ eigenfunc.}}$

Equivalent to diagonalizing a symm. matrix.