

The Rank of a Matrix

Lecture 16

February 14, 2007

Definition

Let $A \in M_{m \times n}(F)$. The **rank** of A , denoted $\text{rank}(A)$ is defined to be the rank of the linear transformation $L_A : F^n \rightarrow F^m$.

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- If $T : V \rightarrow W$ is a linear transformation and β and γ are ordered bases for V and W , then $\text{rank}(T) = \text{rank}([T]_{\beta}^{\gamma})$.

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- 1 $\text{rank}(AQ) = \text{rank}(A) = \text{rank}(PA)$.
- 2 $\text{rank}(PAQ) = \text{rank}(A)$.
- 3 *Elementary row and column operations on a matrix are rank-preserving.*

How to Compute the Rank of a Matrix?

Theorem

The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by its columns.

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Theorem

Let A be an $m \times n$ matrix of rank r . Then $r \leq n$ and $r \leq m$, and, by means of a finite number of elementary row and columns operations, A can be transformed into the matrix

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix},$$

where $0_1, 0_2$, and 0_3 are zero matrices.

Corollary

Let A be an $m \times n$ matrix of rank r . Then there exist invertible matrices B and C of sizes $m \times m$ and $n \times n$, respectively, such that $D = BAC$, where

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix}.$$

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- 1 $\text{rank}(A^t) = \text{rank}(A)$.
- 2 The rank of any matrix equals the maximum number of its linearly independent rows.
- 3 The rows and columns of any matrix generate subspaces of the same dimension, equal to the rank of the matrix.

Corollary

Every invertible matrix is a product of elementary matrices.