

Vibrating Drum

Math 23, Fall 2009

December 1, 2009

Wave equation

Wave equation in \mathbb{R}^2

$$\frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$u = u(x, y, t)$ transverse displacement
 $v =$ wave propagation speed.

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$$\frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = \Delta u$$

Laplace's operator

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \left(+ \frac{\partial^2 u}{\partial z^2} \right)$$

Vibrating circular drum: **Boundary Value Problem**

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PDE: Wave equation $\frac{1}{v^2} u_{tt} = u_{xx} + u_{yy}$

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Domain: $D = \{(x, y, t) \mid x^2 + y^2 < 1, t > 0\}$.

Boundary values:

$$\begin{aligned}u(x, y, t) &= 0 \quad \text{for } x^2 + y^2 = 1, t > 0, \\u(x, y, 0) &= f(x, y)\end{aligned}$$

Circular drum \Rightarrow Polar Coordinates

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$r \geq 0, \quad 0 \leq \theta < 2\pi$$

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Laplace's operator in polar coordinates

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r}$$

BVP in polar coordinates

Wave equation for $u = u(r, t)$ in polar coordinates

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Boundary conditions for vibrating drum

$$u(1, t) = 0, \quad t \geq 0$$

$$u(r, 0) = f(r), \quad 0 \leq r \leq 1$$

Separation of variables

Assume

$$u(r, t) = R(r)T(t)$$

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Wave equation for $u = RT$

$$\frac{1}{v^2}RT'' = R''T + \frac{1}{r}R'T$$

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Obtain two ODEs

$$T'' + \lambda^2 v^2 T = 0 \Rightarrow T = c_1 \cos(\lambda vt) + c_2 \sin(\lambda vt)$$

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$$T'' + \lambda^2 v^2 T = 0 \Rightarrow T = c_1 \cos(\lambda vt) + c_2 \sin(\lambda vt)$$

$$r^2 R'' + rR' + \lambda^2 r^2 R = 0, \quad R(1) = 0$$

Eigenvalue problem

ODE for R

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Change of variables $s = \lambda r$, $R(r) = R(s/\lambda) = \tilde{R}(s)$.

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$$s^2 \tilde{R}'' + s\tilde{R}' + s^2 \tilde{R} = 0, \quad \tilde{R}(\lambda) = 0$$

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Bessel's equation of order zero ($\nu = 0$); solution $\tilde{R}(s) = J_0(s)$

$$R(r) = J_0(\lambda r)$$

Eigenvalue problem

ODE for R

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$$R(r) = J_0(\lambda r)$$

Eigenvalue λ is obtained from boundary condition

$$R(1) = J_0(\lambda) = 0$$

Radially symmetric modes

Solution of BVP

$$R(r) = J_0(\lambda r)$$

$$T(t) = \cos(\lambda vt)$$

Mode $0k$

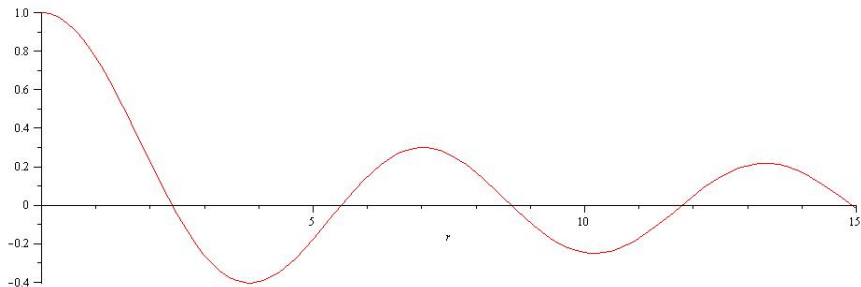
$$u(r, t) = J_0(\lambda_{0k} r) \cos(\lambda_{0k} vt)$$

where λ_{0k} is the k -th zero of $J_0(\lambda) = 0$.

Eigenvalues

Eigenvalues $J_0(\lambda) = 0$

$\lambda_{01} = 2.4048\dots$, $\lambda_{02} = 5.5201\dots$, $\lambda_{03} = 8.6537\dots$,
 $\lambda_{04} = 11.7915\dots$, $\lambda_{05} = 14.9309\dots$, etc.

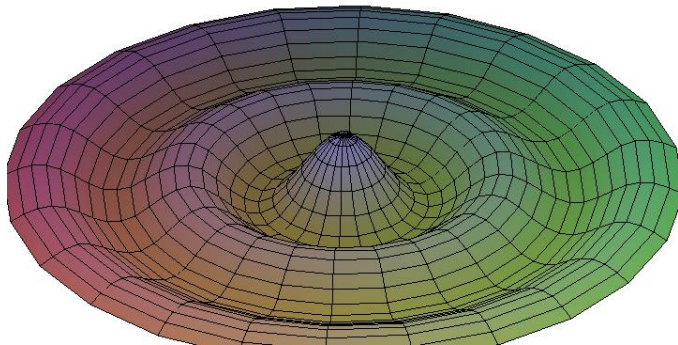


Radially symmetric vibration

Mode 05

$$u(r, \theta, t) = J_0(\lambda_{05} r) \cos(\lambda_{05} vt)$$

Angular frequency = $\lambda_{05} v$, proportional to $\lambda_{05} = 14.9309\dots$
z_cylindrical



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Separation of variables:

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t).$$

Three ODEs, two eigenvalue problems (for R and Θ)

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Three ODEs, two eigenvalue problems (for R and Θ)

Solutions with eigenvalues n, λ

$$\Theta = \cos(n\theta), \quad n = 0, 1, 2, 3, \dots$$

$$R = J_n(\lambda r), \quad \text{Bessel function of order } n$$

$$T = \cos(\lambda vt)$$

$$u(r, \theta, t) = J_n(\lambda r) \cos(n\theta) \cos(\lambda vt)$$

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$$T = \cos(\lambda vt)$$

$$u(r, \theta, t) = J_n(\lambda r) \cos(n\theta) \cos(\lambda vt)$$

Eigenvalue $\lambda = \lambda_{nk}$ is the k -th zero of $J_n(\lambda) = 0$

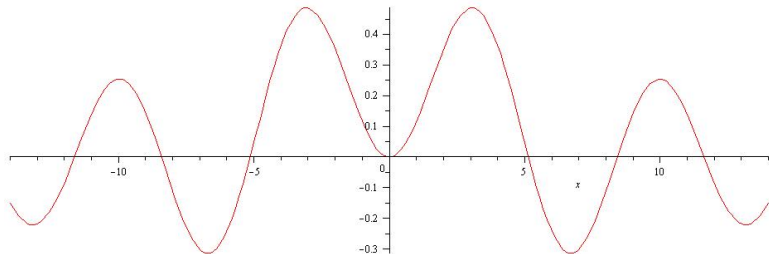
Frequency in mode nk is proportional to λ_{nk} .

Bessel function of order 2

Modes nk with $n = 2$. At time $t = 1$,

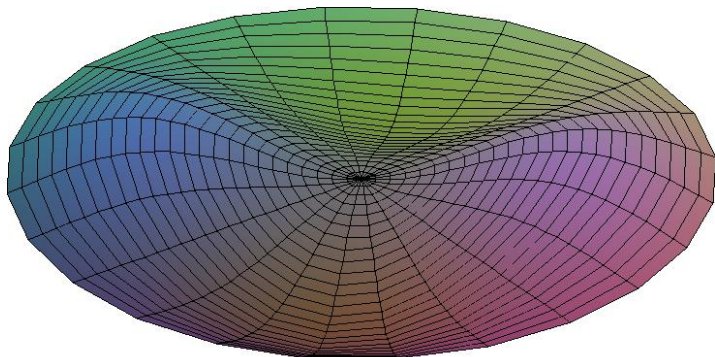
$$u(r, \theta) = J_2(\lambda_{2k}r) \cos(2\theta)$$

$$\lambda_{21} = 5.1356\dots, \lambda_{22} = 8.4172\dots, \text{ etc.}$$

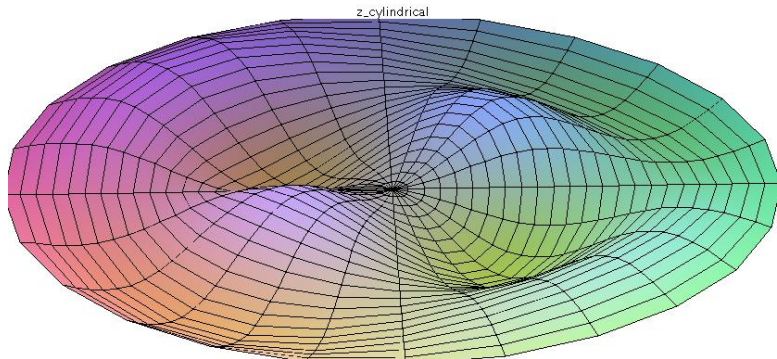


$$u(r, \theta) = J_2(\lambda_{21}r) \cos(2\theta)$$

z_cylindrical

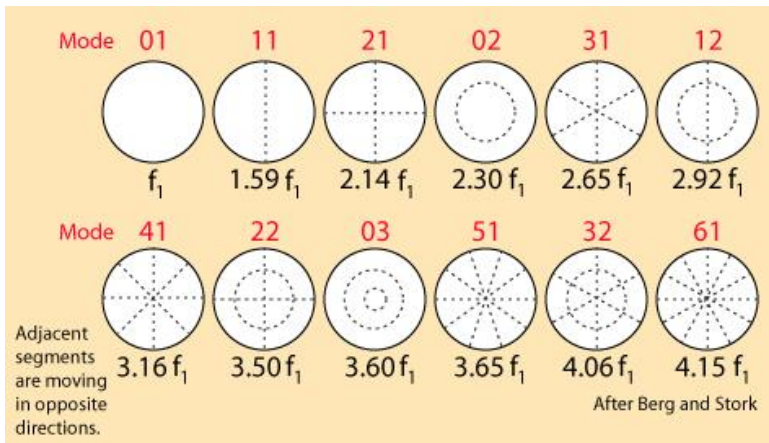


$$u(r, \theta) = J_2(\lambda_{22}r) \cos(2\theta)$$



$$u(r, \theta) = 0 \text{ if } J_2(\lambda_{22}r) = 0 \text{ or } \cos(2\theta) = 0$$

Modes of a vibrating drum



Superposition

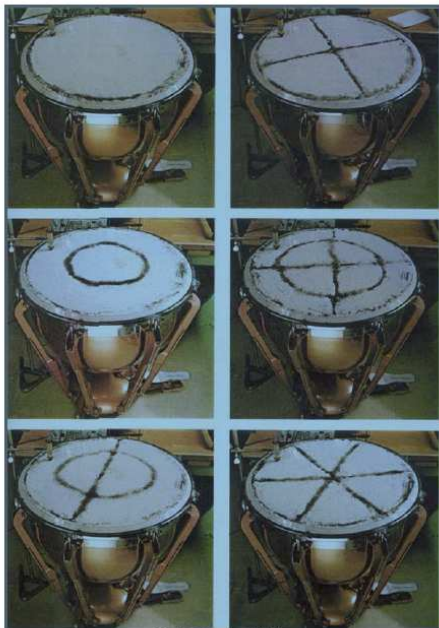
For arbitrary initial value

$$u(r, \theta) = f(r, \theta)$$

vibration of drum = **Superposition** of modes

$$u(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} c_{kn} J_n(\lambda_{nk} r) \cos(n\theta) \cos(\lambda_{nk} vt) + \dots$$

Coefficients c_{kn} are coefficients in “Fourier-Bessel” series of f .



Chladni patterns on a kettle drum