The distribution of mass and zeros for high frequency eigenfunctions on the modular surface

(Dartmouth spectral geometry conference 201

Peter Sarnak

Section 1

1. Report on QUE

"QUE" = Quantum Unique Ergodicity.

(X, g) a compact (or finite area later) RIEMANNIA surface. ϕ Laplace eigenfunction

$$\left.\begin{aligned} &\Delta\phi + \lambda\phi = 0 \\ &\|\phi\|_2 = 1 \end{aligned}\right\} \text{——} (*)$$

- Investigate the behavior of individual such ϕ_λ's as $\lambda \to \infty$.

- Assume that X has negative curvature so that the corresponding classical mechanics (i.e. the geodesic flow) is ergodic, chaotic, . . . and is well understood. For such can't solve $(*)$ explicitly.

Basic questions:

(A). How do the probability masses

$$\nu_\phi = |\phi(x)|^2 dA(x)$$

behave as $\lambda \to \infty$?

(B) How big can ϕ_λ be, say in L^∞ or L^p, $p > 2$?

(C) What can one say about the nodal lines $V(\phi)$ and nodal domains of ϕ?

Conjectured answers (some might say too bold)

(A) Rudnick-S (93) QUE:

$$\nu_{\phi_\lambda} \to \frac{dA}{\text{Area}(X)} \quad \text{as} \quad \lambda \to \infty.$$

In fact the microlocal lifts μ_{ϕ_λ} to $T_1^*(X)$ (the space of unit cotangents) are also equidistributed w.r.t. the corresponding ~~volum~~ Riemannian volume.

$$\mu_\phi(f) := \langle Op(f)\phi, \phi \rangle$$

f a ψ.d.o of degree 0.

(B) For $p > 2$, X compact

$$\|\phi_\lambda\|_p \ll_{p,\varepsilon} \lambda^{\varepsilon} \quad \text{for any } \varepsilon > 0.$$

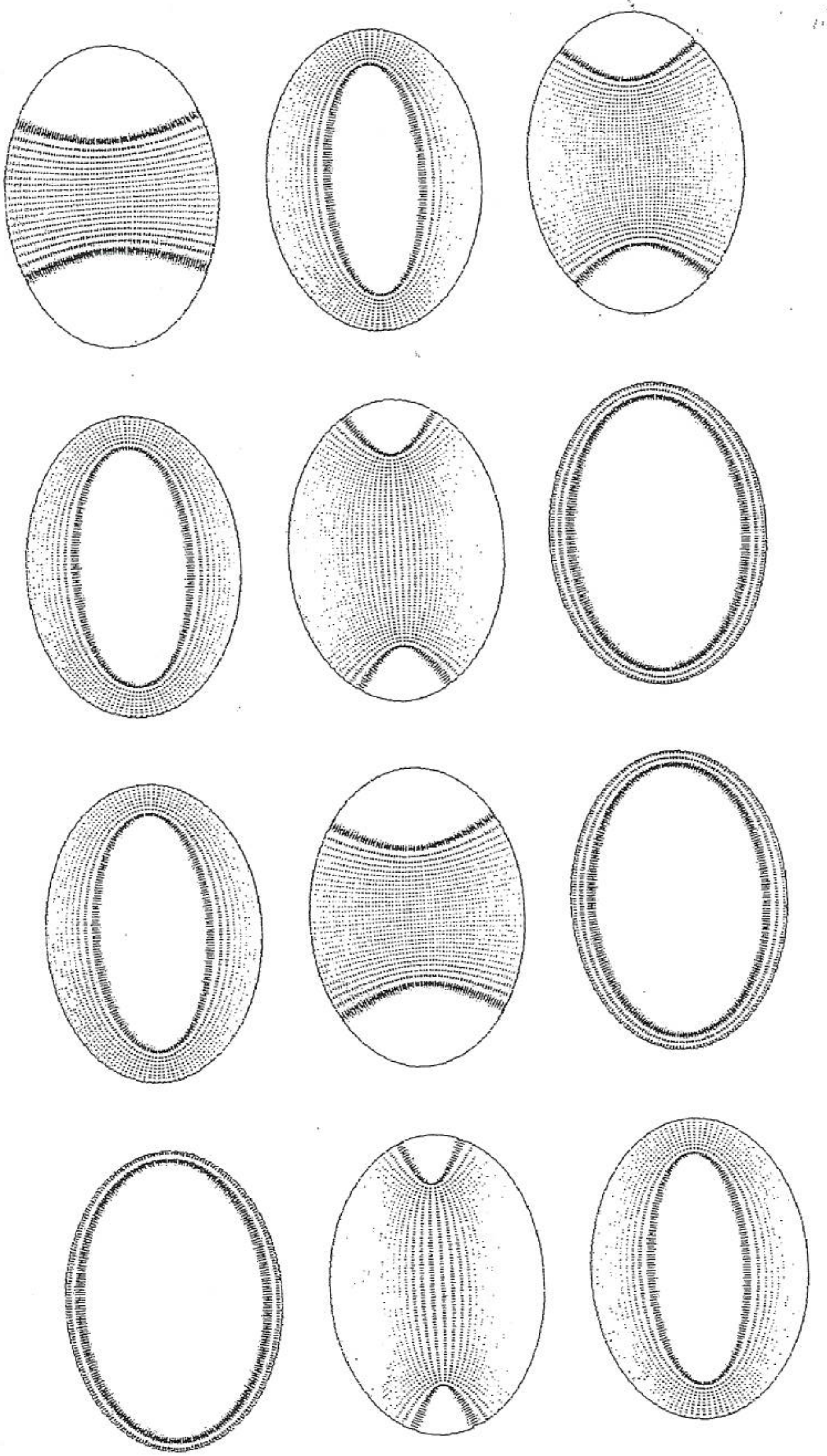
(c) $N(\phi) := \#$ of nodal domains,

Then

$$N(\phi) \sim C_X \lambda \,, \quad \text{as } \lambda \to \infty .$$

This speculation is based on the model of monochromatic random waves (Nazarov - Sodin 09) and even the exact constant C_X has been proposed by Bogomolny and Schmit (02) using a critical percolation model.

What follows are profiles of eigenfunctions on various domains as indicated (all computed numerically)



The case of an <u>integrable</u> billiard.

The densities are of $|\phi|^2 dA$ for eigenfns with $n \simeq 5600$. These concentrate on projections of invariant tori.

Figure by A. BARNETT.

Same as on page 5 but for the ergodic stadium and BARNETT billiard.

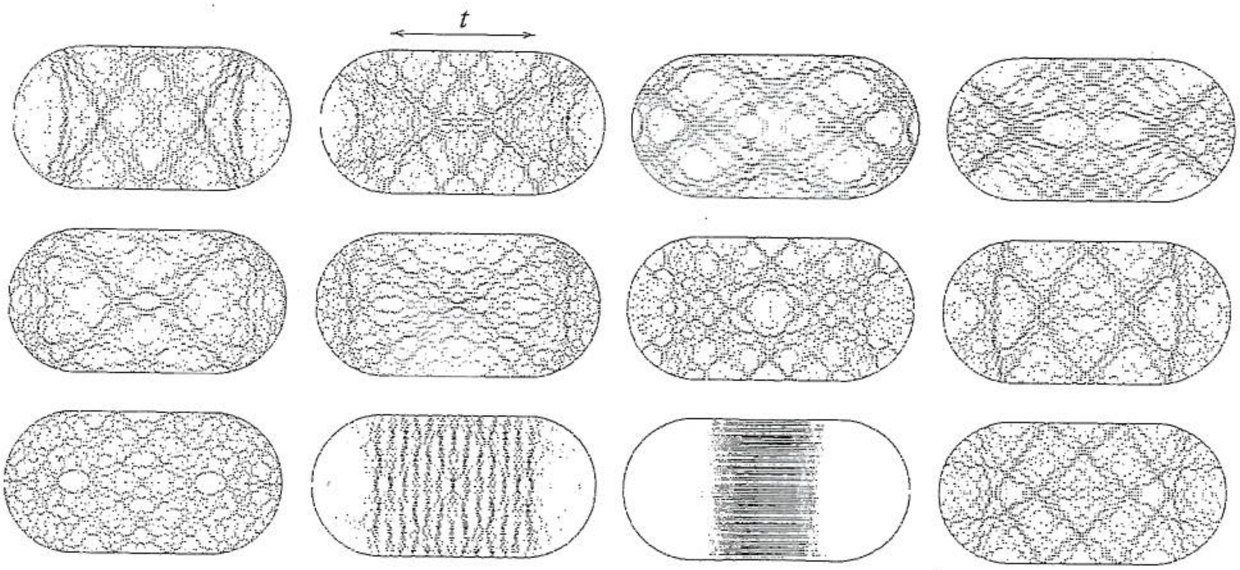


Figure 1S

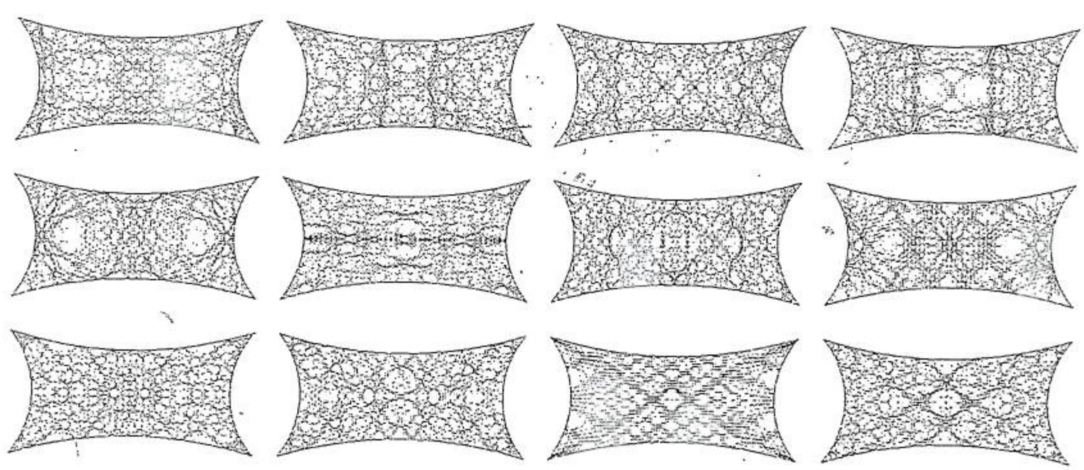
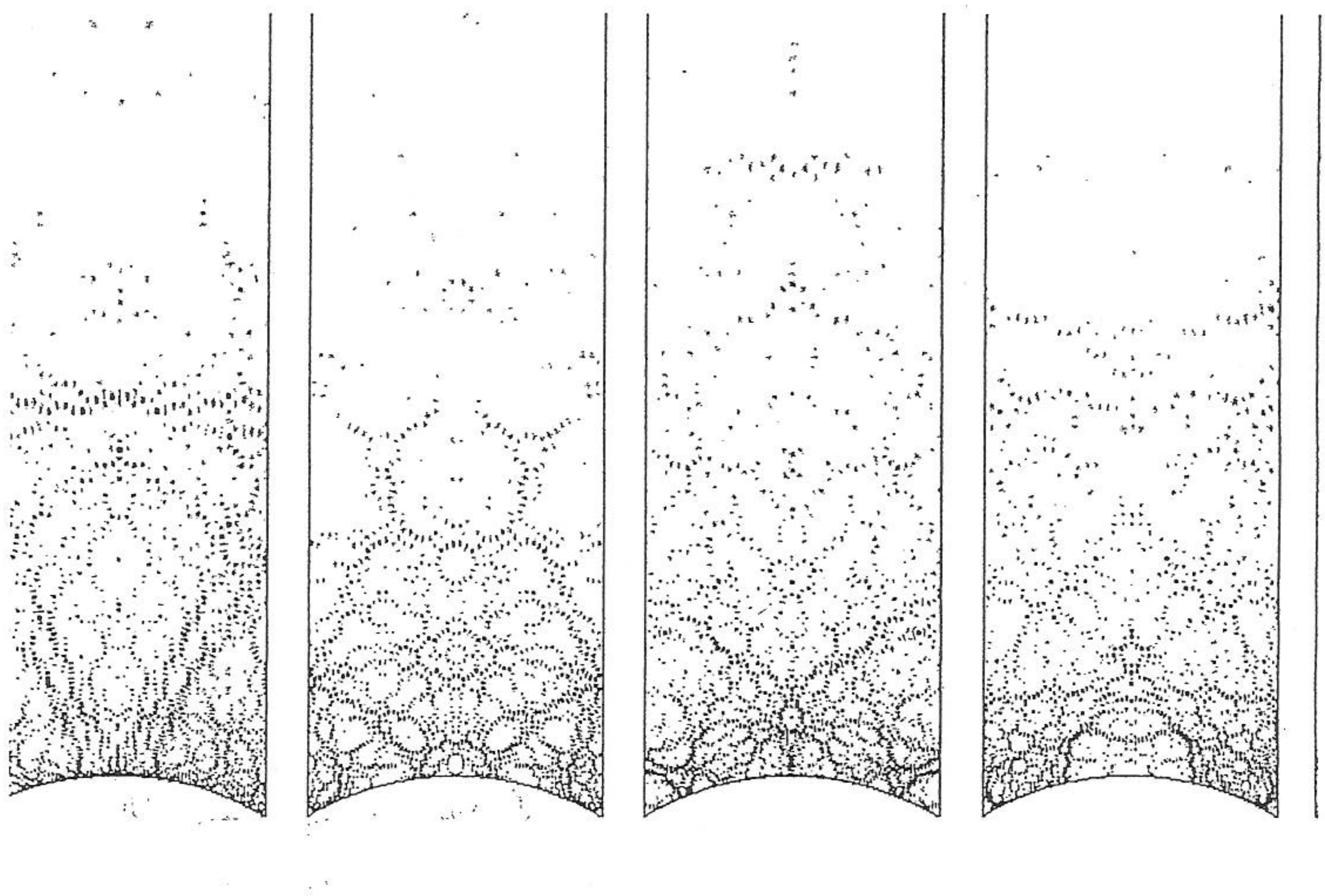


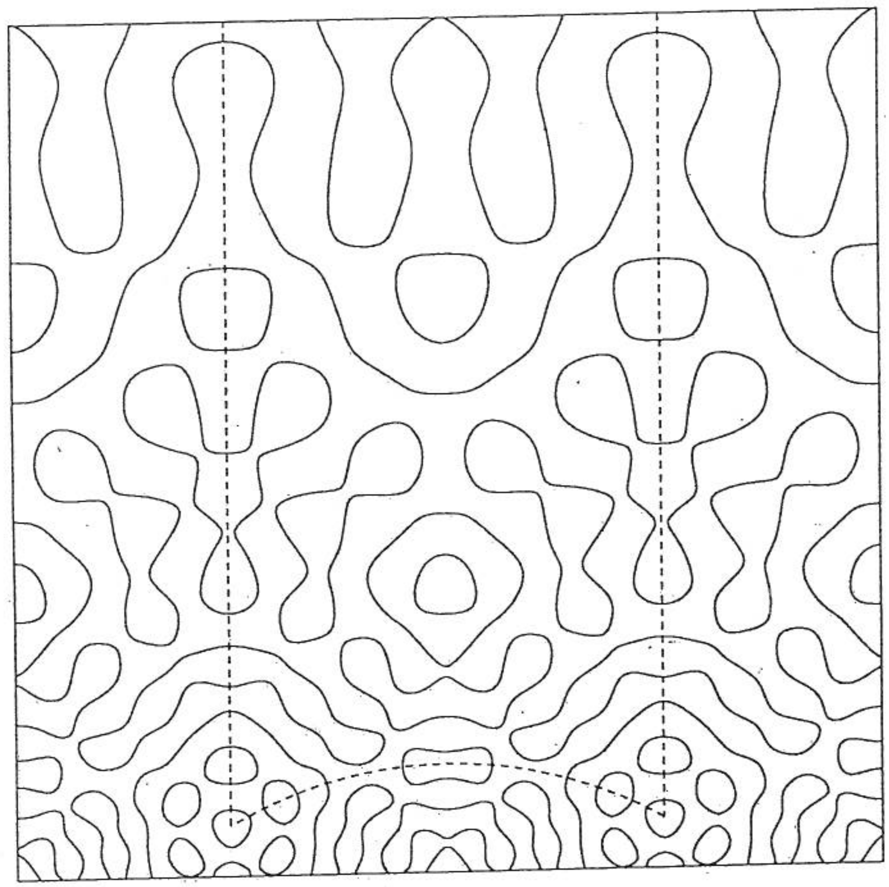
Figure 1B

A. BARNETT

Computation by H. THEN.

Similar picture to page 5 and 6 but for the modular surface $SL_2(\mathbb{Z})\backslash\mathbb{H}$.

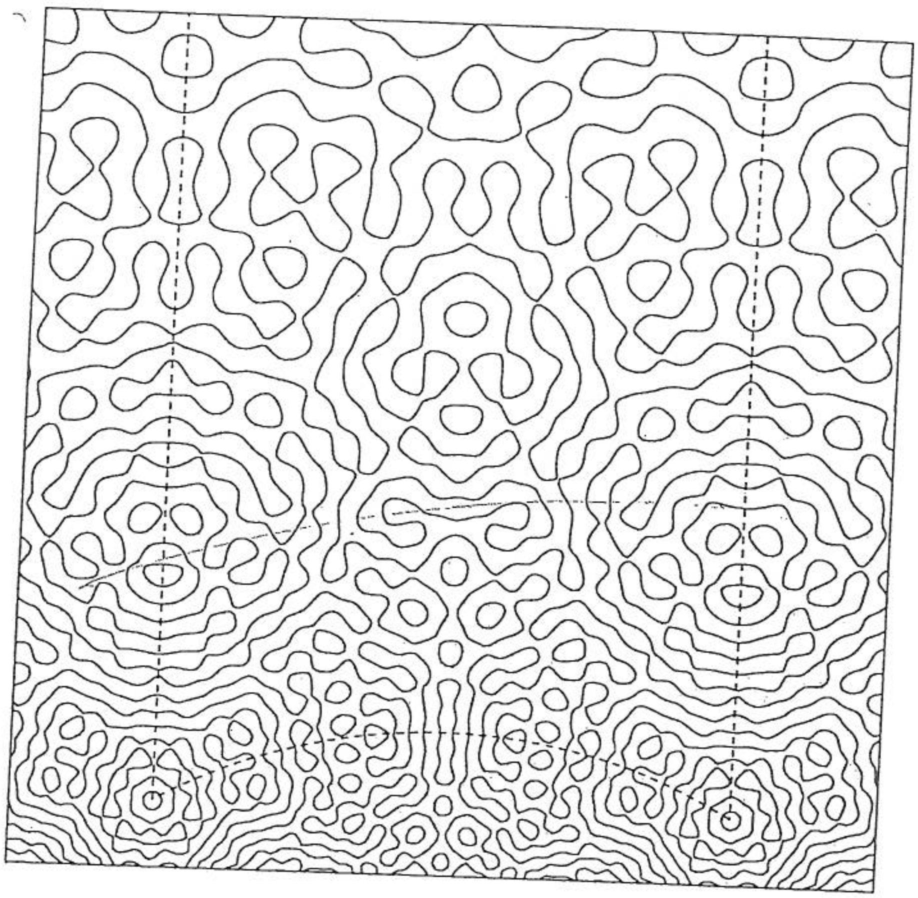




Nodal domains and lines
for $\overset{\text{even}}{\Lambda}\phi$ on the modular surface
$t = 47.926558$,

$\frac{1}{4} + t^2 = \lambda$,

by Hejhal and Rackner



Same as page 8
but
t = 125.523988

(Hejhal - Rackner)

Concerning (A) the only general theorem is:

THEOREM 1 (Anantharaman 08):

Let μ be a quantum limit (i.e. a limit of the μ_{ϕ_λ}'s and ~~[illegible]~~ as such it is known to be geodesic flow g_t invariant), then the entropy, $h(\mu)$ of μ w.r.t g_1, is positive.

This means that μ must be a complicated measure and in particular it cannot be the most singular possibility, that is the arclength measure on an (unstable) periodic geodesic, thus answering this long standing question.

The proof involves a novel combination of the limit of a semi-classical analysis with global hyperbolic dynamics.

A quantitative lower bound for $h(\mu)$ (which is sharp in the analogous context of "quantum cat maps") is due to Rivière (2010):

$$h(\mu) \geq \frac{1}{2} \int_{T_1^*(X)} \lambda_+ \, d\mu$$

where λ_+ is the positive Lyapunov exponent.

All further progress on the basic questions is for arithmetic Riemannian manifolds.

ARITHMETIC QUE:

First restrict to X constant negative curvature $K \equiv -1$. Thus

$$X = \Gamma \backslash \mathbb{H} \quad , \quad \mathbb{H} \text{ the hyperbolic plane}$$

Γ a discrete subgroup of $SL_2(\mathbb{R})$.

Next restrict further:

Γ is 'arithmetic'.

We discuss the prototypical case

$$\Gamma = SL_2(\mathbb{Z})$$

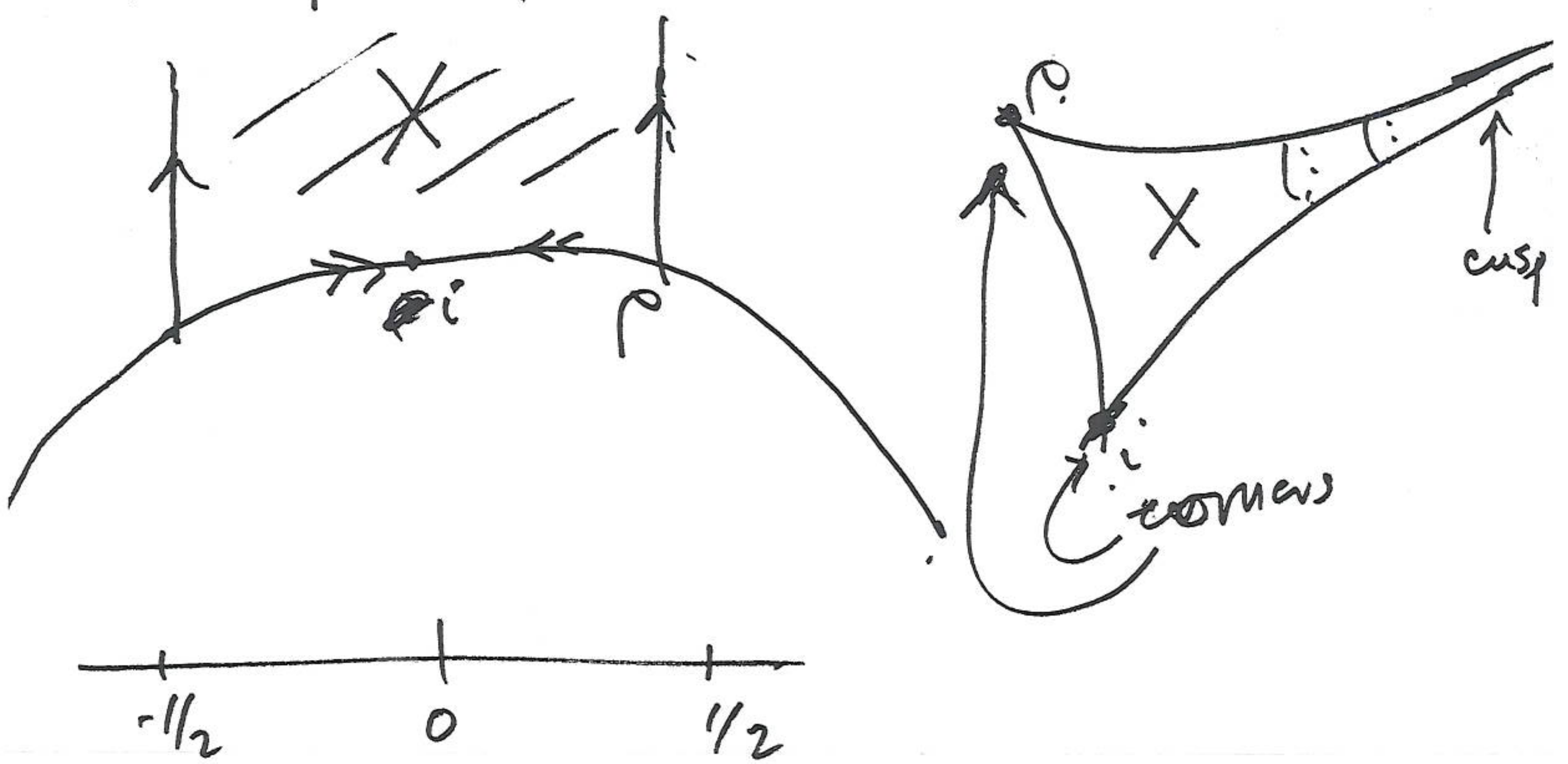
Γ acts on $\mathbb{H}$ by $z \to \frac{az+b}{cz+d}$

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2 .$$

Action is isometric and conformal.

From now on;

$X = \Gamma \backslash \mathbb{H}$, "the modular surface"



Key extra structure :
Algebraic Correspondences (Hurwitz, Hecke, ..)

For $n \geq 1$, $T_n : L^2(X) \to L^2(X)$

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ b \bmod d}} f\left(\frac{az+b}{d}\right).$$

$$T_n T_m = \sum_{d \mid (n,m)} T_{nm/d^2}.$$

$\{T_n, \Delta\}$ are a commutative family of self adjoint operators.

All eigenfunctions ϕ are assumed to be eigenfunctions of this Hecke algebra.

We can certainly choose such an orthonormal basis and probably it is automatic since the λ_ϕ's appear to be simple.

· $n \geq 1$, $T_n \phi = \lambda_\phi(n)\, \phi$

There is also continuous spectrum coming from Eisenstein series which is important but we supress it. The above are called Maass(Hecke) forms.

Concerning Question B for X:

THEOREM 2 (Iwaniec-S 94):
ϕ_λ a Maass form then

$$\|\phi_\lambda\|_\infty \ll \lambda^{5/24}.$$

Note: The (general manifold) local or "convexity" bound for eigenfunctions is $\|\phi_\lambda\|_\infty \ll \lambda^{1/4}$.

THEOREM 3 (WATSON-S 2002).

$$\|\phi_\lambda\|_4 \underset{\varepsilon}{\ll} \lambda^{\varepsilon} \quad \text{for any } \varepsilon > 0.$$

(the proof assumes the Ramanujan-Selberg conjectures for GL_2)

Note these eigenfunctions do get quite large and in Conjecture B and $p = \infty$ we cannot replace λ^{ε} by $\log \lambda$ or even $\log \lambda$ to a big power, in view of:

<u>THEOREM 4:</u> (D. Milicevic 07):

For a fixed compact part K of X (with nonempty interior)

$$\| \phi_{\lambda_j} |_K \|_\infty \gg \exp\left(\frac{\sqrt{\log \lambda_j}}{\log \log \lambda_j} \right)$$

for a subsequence $j \to \infty$ (D.M.J. to appear 2010.)

<u>THEOREM 5</u> (E. Lindenstrauss 06, Soundararajan 09):

QUE holds for X.

[For the continuous spectrum this is due to W. Luo - S and D. Jakobson in the 90's]

HOLOMORPHIC (HECKE) QUE:

$$X = \Gamma\backslash\mathbb{H}, \quad \Gamma = SL_2(\mathbb{Z})$$

k even integer.

$S_k(\Gamma)$ = space of holomorphic cusp forms of weight k;

$$f(\gamma z) = (cz+d)^k f(z), \quad f(\infty)=0$$
$$\gamma = \begin{bmatrix} * & * \\ c & d \end{bmatrix} \in \Gamma,$$
f holomorphic on $\mathbb{H}$.

- $\dim S_k(\Gamma) = \frac{k}{12} + O(1)$ (Riemann-Roch)

$k \to \infty$ is a semiclassical limit in geometric quantization, f's are holomorphic sections of the tensor power $\mathcal{L}^{\otimes k/2}$ of the canonical line $\mathcal{L}$ over X.

We assume that f is a Hecke eigenform:

$$T_n f = \lambda_f(n) f$$

which in terms of a Fourier expansion $\Leftrightarrow$

$$f(z) = \sum_{n=1}^{\infty} n^{(k-1)/2} \lambda_f(n) e^{2\pi i n z}.$$

Holomorphic QUE conjecture (Rudnick-S)

Set

$$\nu_f := |f(z)|^2 y^k \, dA(z), \quad \text{normalized to be}$$

a probability density on X, then

$$\nu_f \to \frac{dA}{\mathrm{Area}(X)} \quad \text{as } k \to \infty.$$

Since $\dim S_k$ is so large, holomorphic QUE is not ~~even~~ valid for all members of S_k and the point of the above is that the Hecke eigencondition suffices to ensure the equidistribution of mass.

• There is no known useful microlocal lift of ν_f to a bigger space so that holomorphic QUE is concerned only with the ν_f's on X.

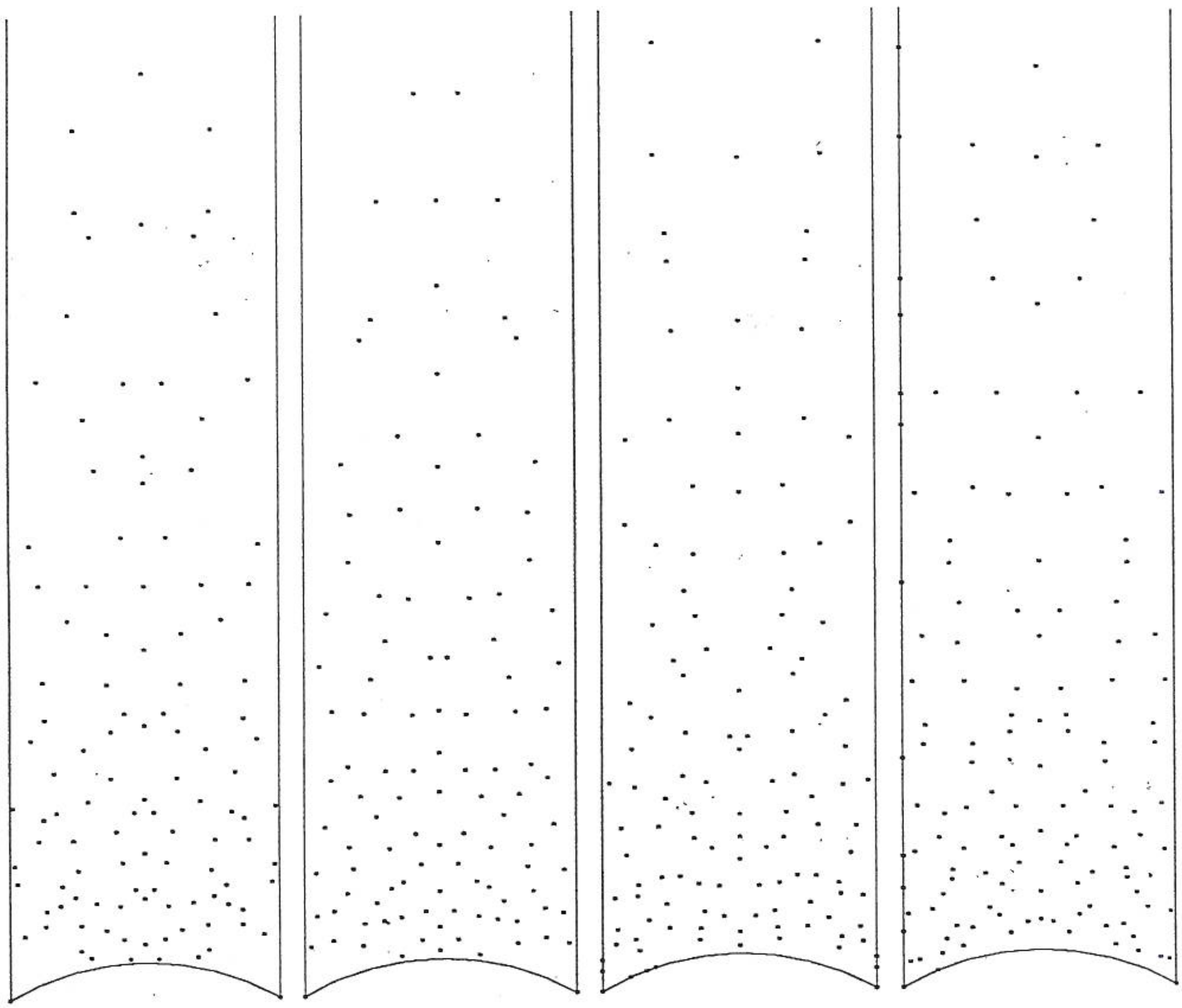
<u>THEOREM 6:</u> (Holowinsky-Soundararajan 09)

Holomorphic (Hecke) QUE is true for $X = SL_2(\mathbb{Z})\backslash\mathbb{H}$.

A formal consequence (with a nice proof!) of the equidistribution of the ν_f's was observed by Nonnenmacher-Voros, Schiffman–Zelditch, and Rudnick:

<u>Corollary 7:</u>

The $\frac{k}{12} + O(1)$ zeros of each such Hecke eigenform f become equidistributed in X w.r.t. $dA/\mathrm{Area}(X)$ as $k \to \infty$.



zeros of holomorphic hecke cusp forms weight $k \cong 2000$ on X
by F. Stromberg

Outline of proofs:

(α) The Maass Case:

Lindenstrauss uses 'measure rigidity' from higher rank homogeneous dynamics. While the fact that the microlocal quantum limit μ on $T_1(X)$ is g_t invariant ~~this~~ does not limit μ very much, the situation for higher rank invariance is very different.

We explain the feature when X is higher rank. For example

$$X = \Gamma \backslash \mathbb{H} \times \mathbb{H}$$

Where Γ is an irreducible lattice in $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ coming from an arithmetic group:

Consider $SL_2(\mathbb{Z}[\sqrt{2}])$ and let $'$ denote the Galois conjugation of $\mathbb{Q}(\sqrt{2})$.

$SL_2(\mathbb{Z}[\sqrt{2}])$ yields a lattice Γ in G by the embedding

$$SL_2(\mathbb{Z}[\sqrt{2}]) \ni \gamma \longrightarrow (\gamma, \gamma') \in G.$$

X is a Hilbert modular surface (as a complex manifold).

$\Delta_{z_1}, \Delta_{z_2}$ the Laplacians in each of $(z_1, z_2) \in \mathbb{H} \times \mathbb{H}$.

$$\Delta = \Delta_{z_1} + \Delta_{z_2}, \quad \Delta_{z_1}, \Delta_{z_2} \text{ commute}$$

We assume that ϕ is an eigenfunction of each of $\Delta_{z_1}, \Delta_{z_2}$ simultaneously.

$$\nu_\phi = |\phi(z_1, z_2)|^2 dA(z_1)\, dA(z_2)$$

is now a density on X (probability density).

One can define a natural micro-local lift μ_ϕ of ν_ϕ to $\Gamma\backslash G$.

Note that this lift is to a smaller space than $T_1^*(X)$. The ~~a~~ crucial point is that any "quantum limit" μ on $\Gamma\backslash G$, that is a weak limit of the μ_ϕ's is now invariant under a bigger flow: that is the full diagonal flow A of $\Gamma\backslash G$ given by:

$$\Gamma g \longmapsto \Gamma g\left(\begin{pmatrix} a_1 & \\ & a_1^{-1}\end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^{-1}\end{pmatrix}\right)$$

where $a_1, a_2 \in \mathbb{R}^*$.

That such A-invariant measures on $\Gamma\backslash G$ should be rigid goes back to Furstenberg. He conjectured that a probability measure μ on $[0,1]$ invariant under both $x \to 2x$ and $x \to 3x$ must be of a certain simple form. Assuming that μ is ergodic and of positive entropy Rudolf showed that ~~[illegible]~~ $\mu = dx$. Katok-Einsiedler and Lindenstrauss have established similar results in the setting of higher rank spaces $\Gamma\backslash G$.

THEOREM 8 (E. Lindenstrauss):

Let X be the Hilbert modular space and μ an A-invariant measure all of whose ergodic components are of positive ~~measure~~ entropy, then $\mu = dg_1 dg_2$.

In separate works extending arguments of Rudnick-S, Bourgain and Lindenstrauss prove that if the ϕ's are also Hecke eigenforms ~~[illegible]~~ then in fact $h(\mu) > 0$ for every ergodic component of μ. This together with an argument of Soundararajan concerning escape of mass (that is it can't happen) establishes the QUE conjecture for the Hilbert modular variety X. S. Brooks and E. Lindenstrauss have apparently established this without assuming that ϕ is an eigenfunction of the Hecke operators.

The case of the modular surface X, ie Theorem 5 is more difficult. Lindenstrauss uses that ϕ is a Hecke eigenform for a fixed prime p and develops a measure rigidity theory in the space

$$\widetilde{\Gamma} \backslash SL_2(\mathbb{R}) \times PGL_2(\mathbb{Q}_p)$$

where $\widetilde{\Gamma}$ is the familiar lattice in the product. This time one does not have invariance under the diagonal in the second factor but only a certain recurrence which he shows is sufficient to develop the rigidity phenomenon.

- These ideas extend to higher rank symmetric spaces, assuming that the ϕ's are eigenfns of the full ring of commuting differential operators allows one to establish suitable forms of QUE ~~for these~~. (Silberman - Venkatesh 08).

(β) <u>The holomorphic QUE case:</u>

The techniques that lead to the proof of Theorem 6 are of a very different nature, involving modular forms their L-functions and number theory methods. We review the basics:

<u>Zeta and L-functions:</u>

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p} (1-p^{-s})^{-1}$$

<u>Riemann:</u>

The completed function Λ;

~~satisfies~~ $\Lambda(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s)$

Satisfies

$$\Lambda(1-s) = \Lambda(s)$$ functional equation

and is entire except for ~~possible~~ simple poles at $s=0$ and 1.

- <u>R-H</u> ; <u>Riemann's Hypothesis</u>:
  All the zeros of $\Lambda(s)$ are on $\mathrm{Re}(s) = \frac{1}{2}$.

<u>Bounds</u> (turn out to be already very powerful):

- <u>L-H</u> <u>Lindelof Hypothesis</u>:

for $\varepsilon > 0$, $\zeta(\frac{1}{2} + it) = O_\varepsilon\left((|t|+1)^\varepsilon \right)$, $t \in \mathbb{R}$.

- $RH \Rightarrow LH$ (Pf: apply complex interpolation to $\log \zeta(s)$)

<u>Convexity</u>: (complex interpolation + funct. eq^n)

$$\zeta(\tfrac{1}{2}+it) = O(|t|^{1/4}).$$

$|t|$ = "conductor".

The first subconvex result:

Weyl (1917):

$$\zeta(\tfrac{1}{2}+it) = O(|t|^{1/6}) .$$

(i.e. a power saving in the exponent - over the convex bound).

Our ϕ's and f's (Maass and holomorphic) are all automorphic forms and give rise to L-functions generalizing the Riemann Zeta function. For many this is the reason one studies such automorphic forms as they are the source of all 'L-functions'.

The connection between QUE and certain L-functions was explicated in Watson's thesis (2000)

<u>Watson's Formula:</u>

ϕ_1, ϕ_2, ϕ_3 three Hecke-Maass cusp forms on X:

$$\left| \int_X \phi_1(z)\, \phi_2(z)\, \phi_3(z)\, dA(z) \right|^2 = C_X \frac{\Lambda(\frac{1}{2}, \phi_1 \times \phi_2 \times \phi_3)}{\Lambda(1, \mathrm{sym}^2\phi_1)\Lambda(1, \mathrm{sym}^2\phi_2)\Lambda(1, \mathrm{sym}^2\phi_3)}$$

Here the numerator is the value at $s = \frac{1}{2}$ of the Rankin triple product L-function associated to $\phi_1 \times \phi_2 \times \phi_3$. It is known to have an analytic continuation and functional equation thanks to the work of GARRETT. The denominators involve the values at $s=1$ of symmetric square L-fns

Applying Watson with $\phi_1 = \phi_2 = \phi_\lambda$ and with ϕ_3 fixed and analysing the archimedian factors using Stirling's formula for the Gamma fn yields (with $\lambda = \frac{1}{4} + t^2$)

$$\text{L.H.S.} = \left| \nu_{\phi_\lambda}(\phi_3) \right|^2 \approx \frac{|t|^{-1} L(\frac{1}{2}, \phi_1 \times \phi_2 \times \phi_3)}{L(1, \operatorname{sym}^2 \phi_\lambda)^2}.$$

QUE demands and is essentially equivalent to L.H.S $\to 0$ as $\lambda \to \infty$. So one is faced with estimating L-functions at $s = 1/2$ and $s = 1$. At $s=1$ everything is controled by $(\log t)^{\pm 1}$ by arguments going back to Hadamard and de la Vallee Poussin in their proof of the prime number theorem.

The analytic conductor of $\phi_\lambda \times \phi_\lambda \times \phi_3$ which measures the complexity of this form is of size $|t|^4$, so that the general convexity bound yields

$$L(\tfrac{1}{2}, \phi \times \phi \times \phi_3) \ll (\mathrm{cond})^{1/4} = |t| .$$

We see that this just fails to give what we want but that a subconvex bound would do.

· That is subconvexity gives a power saving in the decay of $\nu_{\phi_\lambda}(\phi_3)$ as $\lambda \to \infty$, which we will call quantitative CQUE.

Note that L-H for $L(s, \phi \times \phi \times \phi_3)$ yields the decay of $|t|^{-1/2+\varepsilon}$ for $\nu_\phi(\phi_3)$ which one can show is the optimal rate of equidistribution.

It is for applications such as this That much effort has been put into establishing subconvex estimates for general L-functions. Indeed QUE for the continuous spectrum on X and for special forms ϕ called 'dihedral' is established exactly in this way. To date however subconvexity for the family $L(\frac{1}{2}, \phi \times \phi \times \phi_3)$ is not known.

Note: Watson's formula applies to holomorphic Hecke forms and subconvexity for the corresponding triple product L-functions implies a quantitative holomorphic QUE.

Weak Subconvexity (Soundararajan 09):

π an automorphic cusp form on GL_n, n-fixed

$L(s,\pi)$ its L-function

$C(\pi)$ its (analytic) conductor.

convexity:

$$L(\tfrac{1}{2},\pi) \ll C(\pi)^{1/4} .$$

Definition (Sound): weak subconvexity if there is $\varepsilon > 0$ s.t.

$$L(\tfrac{1}{2},\pi) \ll C(\pi)^{1/4} \Big/ (\log C(\pi))^{\varepsilon} .$$

<u>Note</u>: This estimate alone is not enough to prove QUE because of the denominator factors $L(1, \mathrm{sym}^2\phi)$.

<u>THEOREM 9</u> (Soundararajan 09):
If π on GL_n satisfies the generalized Ramanujan Conjectures then $L(\frac{1}{2}, \pi)$ satisfies a weak subconvex bound.

The proof is based on a far-reaching extension of the theory of mean values of multiplicative functions (Wirsing, Halasz ...).
The method can only yield a small power of $\log C(\pi)$.

Now for $f \in S_k(\Gamma)$ holomorphic modular hecke cusp form ~~ ~~ is known to satisfy the Ramanujan Conjectures ! (Deligne 74): i.e.

$$\boxed{|\lambda_f(n)| \leq d(n) = \sum_{d|n} 1} .$$

[Note the The Ramanujan Conjecture ~~are~~ not known for Maass f]

<u>Cor 10 (Soundararajan).</u>

$L(\frac{1}{2}, f \times f \times \phi_3)$ satisfies ~~ ~~ a weak subconvex bound.

<u>Cor 11 (Sound):</u>

Holomorphic QUE is true for all but $O_\varepsilon(k^\varepsilon)$ of the $\frac{k}{12} + O(1)$ hecke forms in $S_k(\Gamma)$.

Another approach to QUE is due to Luo-S (2003) and is based on 'shifted convolutions'

QUE (hol) for f $\iff$ for $h \in \mathbb{Z}$ say $h \neq 0$ fixed

$$\sum_{n \leq k} \lambda_f(n)\lambda_f(n+h) = o\left(k\, L(1, \mathrm{sym}^2 f)\right).$$

One expects that there is substantial cancellation in this sum due to the sign changes in $\lambda_f(n)$.

Holowinsky's remarkable idea is to forgo this cancellation and to exploit the feature that the $|\lambda_f(p)|$'s are more often smaller than 1 which is to be expected from the fact that these coefficients follow the Sato-Tate distribution (at least if f is not dihedral but in that case QUE is established) This distribution forces the $|\lambda_f(n)|$'s to be on average of size $1/\log n$. Using this idea and an elementary sieve as well as the Ramanujan Conjectures he shows that

$$\sum_{n \leq k} |\lambda_f(n)\, \lambda_f(n+h)| \ll \frac{k}{(\log k)^2}$$

as long as $L(1, \mathrm{sym}^2 f)$ and $L(1, \mathrm{Sym}^4 f)$ are of order 1.

<u>THEOREM 12:</u> (HOLOWINSKY 09)

Holomorphic QUE is true for all but $O_\varepsilon(k^\varepsilon)$ of the $\frac{k}{12} + O(1)$ hecke forms in $S_k(\Gamma)$.

Remarkably the ~~ex~~ methods of Sound and Holowinsky are sufficiently different that the exceptional sets in their theorems can be shown not to intersect and holomorphic QUE follows!

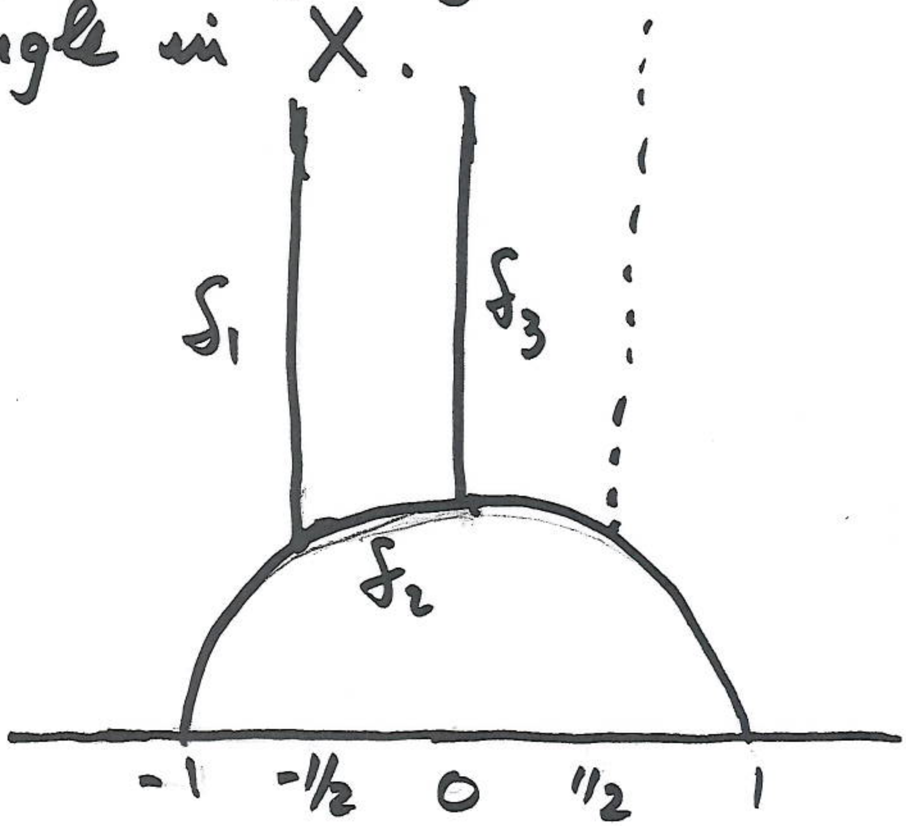
All of these ideas can be extended to holomorphic Hilbert modular forms as is shown by S. Marshall in his 2010 thesis. In particular he shows that the zero divisors of such forms in several complex variables become equidistributed in the Hilbert modular variety as their weight increases. I find this to be a quite striking application of what one might call QUE theory.

Section 2 Nodal and Zero Sets

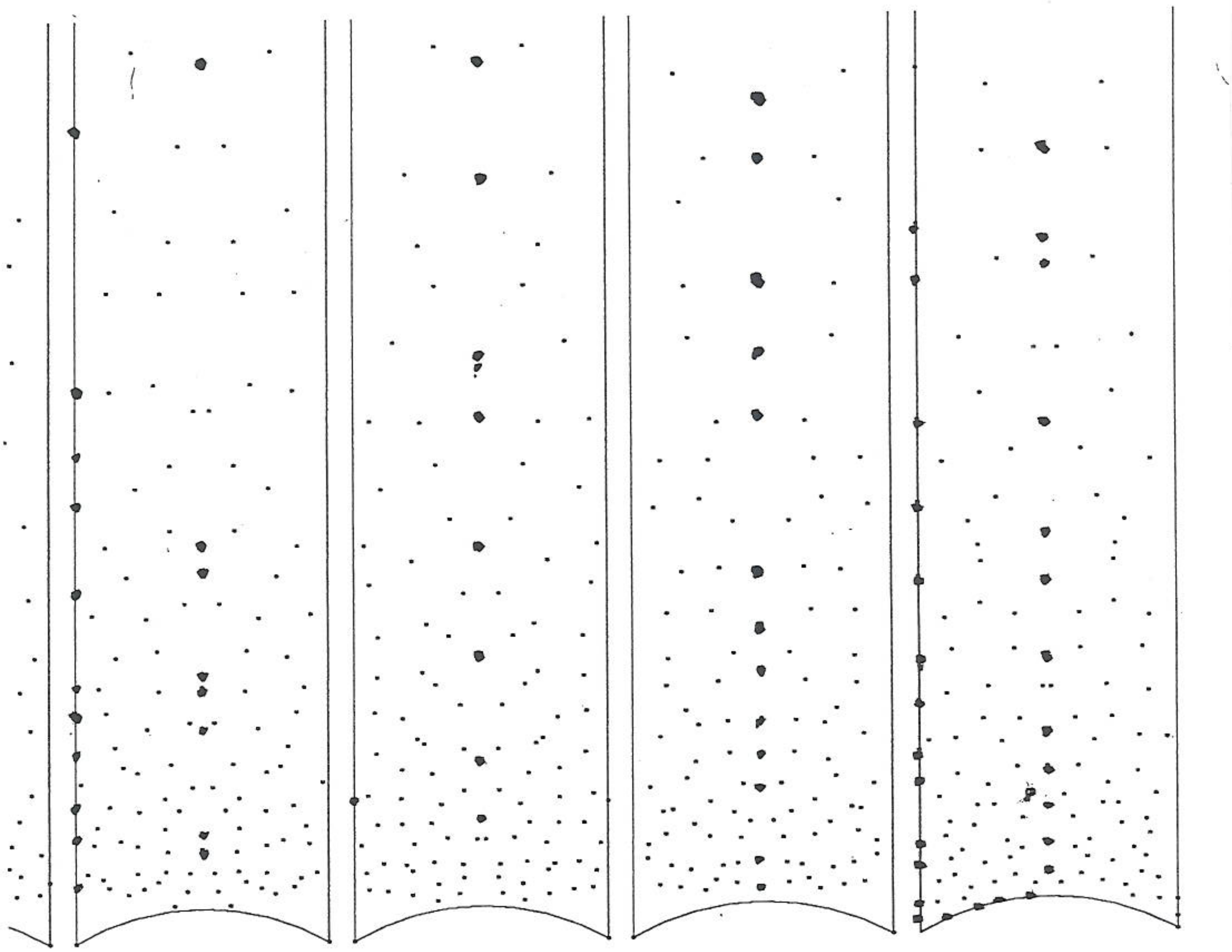
The modular surface carries three global anti-holomorphic involutions;

$$S_1(z) = -\bar{z}\ ,\ S_2(z) = 1-\bar{z}\ ,\ S_3(z) = -1/\bar{z}\ .$$

The geodesic segments $\delta_1, \delta_2, \delta_3$ shown below are fixed pointwise by S_2, S_3, S_1 resp. Let $\delta = \delta_1 \cup \delta_2 \cup \delta_3$ be the corresponding triangle in X.



A holomorphic hecke cusp form f is real on δ_1 and δ_3 from which it follows that the set of zeros of f are invariant under each S_j. We may expect that f has a growing number of zeros on δ.



Same as page 19 but
with zeros on δ highlighted.

THEOREM 13 (GHOSH - S 2010):

Let $N_f(\delta)$ denote the number of zeros of f lying on δ, then for f a hecke cusp form of weight k, $N_f(\delta)$ goes to infinity with k, in fact

$$N_f(\delta) \gg k^{13/56}.$$

Notes: 1) probalistic reasoning suggests that $N_f(\delta) \sim C_X \sqrt{k} \log k$ as $k \to \infty$.

2). The lower bound in Theorem 13 should hold for each side δ_j separately but we don't have a proof of that.

Idea of proof:

- f is real on $\delta_1 \cup \delta_3$, using the Fourier expansion (page 17) we look for sign changes of f on these curves.

- For $\sqrt{k} \le y \le k$ a steepest descent analysis shows that the sign of f on $\delta_1 \cup \delta_2$ is dictated by that of $\lambda_f(m)$ for $y_m \sim c\, k/m$, $1 \le m \le \sqrt{k}$ (if this number is not too small).

- The relation for p prime,

$$\lambda_f(p)^2 = \lambda_f(p^2) - 1$$

shows that one of these coefficients is not small (this has become a standard device used first in the proof of Theorem 2 above).

- A combinatorial analysis together with the fact that almost all intervals $[y, y+\Delta y]$ with $\Delta y = y^{1/14}$ contain a prime (Harman-Watt) leads to the demonstration of many sign changes.

Before discussing the Maass form nodal domains on pages 8+9 we recall some general facts about nodal lines and domains for a compact analytic surface Y. Denote by $N(\phi)$ the number of nodal domains and $\nu(\phi)$ the nodal curve in Y.

(1) (Courant) If ϕ is the n-th eigenfunction counted with multiplicity then

$$N(\phi) \leq n$$

(2) (Donnelly-Fefferman, Bruening smooth case)

$$\sqrt{\lambda} \ll \text{length}(\nu(\phi)) \ll \sqrt{\lambda}$$

(3) $\nu(\phi)$ is $1/\sqrt{\lambda}$ dense in that it meets every ball of radius $1/\sqrt{\lambda}$.

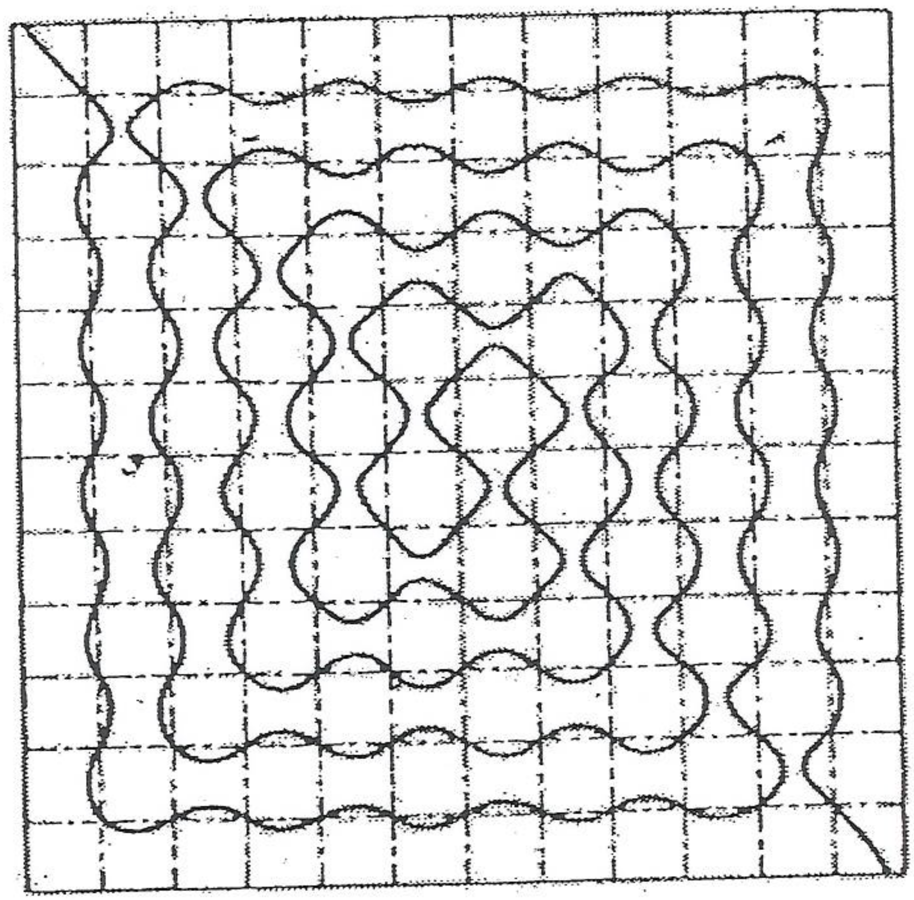
(4) In many ways $\nu(\phi)$ behaves like a projective real plane curve of degree $t = \sqrt{\lambda}$. It would be very interesting to establish the following Bezout like theorem:

If $\alpha \subset Y$ is a real analytic segment then either $\alpha \subset \nu(\phi)$ or

$$\#\,(\alpha \cap \nu(\phi)) \ll t .$$

For planar domains with analytic boundary and ϕ a Neumann eigenfunction Toth and Zelditch establish the above when α is part of the boundary.

Lower bounds for $N(\phi)$ and for the no' of intersections $\nu(\phi)$ with a given analytic segment α are difficult to establish since in general these need not grow. The following example of an eigenfunction on the square with $N = 2$ illustrates the ~~difficult~~ point.



Nodal domain of an eigenfunction on the square, $N(\phi)=2$.

from Courant-Hilbert; Vol I.

Thesis A. Stern Gottingen 1925

Back to $X = SL_2(\mathbb{Z})\backslash\mathbb{H}$ and ϕ a Maass-Hecke eigenform.
All the results stated below are work in progress with A. Ghosh and A. Reznikov.

(4) The role of the triangle $\delta \subset X$:

Proposition 14: Let $\alpha \subset X$ be a geodesic segment and suppose that ϕ (not necessarily a Hecke eigenform for this propn) vanishes on α, then $\alpha \subset \delta$.

For the proof ~~one~~ we use that $\langle \Gamma, R_1 \rangle$ is a maximal discrete subgroup of the isometries of $\mathbb{H}$.

- The Maass-Hecke eigenforms on X break up into even and odd ones. The even ones satisfy Neumann bdry conditions on δ and the odd ones satisfy Dirichlet boundary conditions.

<u>Proposition 15:</u> X the modular surface, ϕ a Maass-Hecke eigenform then

$$N(\phi) \gg t.$$

- The proof of this is a bit of a cheat since it produces this no' of nodal domains high in the cusp where the shape of ϕ simplifies. That is for $t\alpha \le y \le \beta t$ with

$$\tfrac{1}{2} < \beta < \alpha < 1 \quad ,$$

$$\phi(z) = \sum_{n=1}^{\infty} \lambda_\phi(n) y^{1/2} K_{it}(2\pi|n|y) \cos(2\pi n x)$$

the primary contribution is from $n = 1, \ldots$

- The real challange is to produce a growing number of nodal domains in a fixed compact part K of X.

- For α an analytic arc segment in X, let $N_\alpha(\phi)$ be the number of nodal domains of ϕ whose boundary meets α.

To estimate $N_\alpha(\phi)$ we need to understand first the intersections of $\nu(\phi)$ with such arcs α.

THEOREM 1.6: Let γ be a closed horocycle on X then

$$t^{1/12} \ll \#(\nu(\phi) \cap \gamma) \ll t$$

. steps in the proof;

1) a sharp L^2-restriction theorem to γ (see letter to A. Reznikov www.math.princeton.edu/sarnak)

$$1 \ll \|\phi_\gamma\|_2 \ll t^{\varepsilon}$$

2). For $0 \leq \xi < \eta \leq 1$

$$\int_\xi^\eta \phi(x+iy_0)\,dx \ll_{\varepsilon, y_0} t^{-1/2+\varepsilon}$$

(the implied constant depends on ε and y_0 only).

3) $$1 \ll \int_0^1 |\phi(x+iy_0)|^2 dx \leq \|\phi\|_\infty \int_0^1 |\phi(x+iy_0)|\,dx$$

$$\leq \sum_{j=1}^{\nu} \left| \int_{\alpha_j}^{\beta_j} \phi(x+iy_0)\,dx \right| \|\phi\|_\infty$$

fixed sign.

$$\leq \nu\, t^{-1/2+\varepsilon}\, t^{5/12}$$

by Theorem 2 and it is critical that the latter is a "subconvex" bound!

To prove similar results for more general analytic arcs α we need to assume subconvexity for the triple product L-functions $L(s, \phi \times \phi \times \phi_3)$, that is a quantitative form of QUE. We explicate this in the important case that $\alpha \subset \delta$.

THEOREM 17: Assume the Lindelof Hypothesis for the various L-functions above (this is more than we need). Let $\alpha \subset \delta$ be a geodesic segment then for ϕ even (there is an analogous statement for ϕ odd)

$$t^{1/12} \ll \#(\mathcal{V}_\phi \cap \alpha) \ll t .$$

From this it follows from a simple topological/combinatorial argument that there are many nodal domains. In particular the following corollary implies that the number of nodal domains of ϕ in a compact part of X tends to infinity as $t \to \infty$.

<u>Corollary 18:</u> Under the same assumptions as in Theorem 17 we have that for any fixed segment α contained in δ

$$t^{1/12} \ll N_\alpha(\phi) \ll t \quad .$$

We can say nothing about the presumably very many ($\approx t^2$) nodal domains that don't touch δ.

• To end consider a related general problem concerning automorphic L-functions. π ~~are~~ a self-dual automorphic cusp form on GL_n, (such as our ϕ' and f's on X, $n=2$).
~~The~~ Its completed L-function satisfies

completed $= \Lambda(s,\pi)$; $\Lambda(1-s,\pi) = \varepsilon(\pi)\Lambda(s,\pi)$
$\varepsilon(\pi) = \pm 1$ and $\Lambda(s,\pi)$ is real for $s \in \mathbb{R}$.

<u>Definition:</u> Λ is positive definite if the function $\Lambda(\frac{1}{2}+it,\pi)$ is a positive definite function of t, $t \in \mathbb{R}$ [i.e. for $t_1, t_2, \dots t_\nu \in \mathbb{R}$, $\left(\Lambda\left(\frac{1}{2}+i(t_k-t_\ell)\right)\right)_{\substack{1\le k\le \nu \\ 1\le \ell\le \nu}}$ is a positive definite hermitian matrix]

<u>Note 1:</u> If Λ is positive definite then $\Lambda(s)$ has no real zeros.

2) For $\pi = \phi$ or f on X this positive definiteness is equivalent to ϕ (resp f) having no zeros on δ_3, (of course we restrict to $\varepsilon(\pi) = 1$).

For π's of small conductor apparently many of the Λ's are positive definite. The most notable example is the Riemann zeta function where $\frac{s(1-s)}{2}\Lambda(s)$ is positive definite as noted by Riemann himself. The vast majority of Dirichlet L-functions $\Lambda(s, \chi)$ with χ quadratic (i.e. selfdual) and of small conductor are positive definite. This phenomenon disappears as the conductor $c(\chi)$ grows as shown by R.C. Baker and H.L. Montgomery (1990) (precisely the percentage of such ~~χ's~~ positive definite χ's tends to zero). E. Bachmat who is interested in such positive definite π's from ~~a~~ his investigations in queuing theory informs me that the majority of elliptic curve L-functions whose ranks are 0) and of conductor about 100,000 are positive definite.

The proof of Proposition 15 shows that there are only finitely many ϕ's on X which are positive definite while for the f's on X, Theorem 13 falls just short of showing this since the zeros are shown to lie in δ rather than δ_3.

The (analytic) conductor $C(\pi)$ of π is a product $C_\infty(\pi)q(\pi)$, where $C_\infty(\pi)$ is the archimedian piece and $q(\pi) \in \mathbb{N}$ is comprised of primes at which π is ramified. For ϕ and f on X, $q(\pi)=1$ and it is $C_\infty(\pi)$ that increases and as we have seen there should be many zeros on δ_3 (at least a small power of $C_\infty(\pi)$).

In the case of Dirichlet characters χ, ($\chi^2 = 1$) and elliptic curve L-functions $c_\infty(\pi)$ is fixed and $q(\pi) \to \infty$. In this case various considerations lead to the expectation that there are far fewer zeros developing on $\delta_1 \hat{q}$. It is thus much more difficult to demonstrate that these zeros must occur. We are led to the simple question

Question: Is the set of π's for which $\Lambda(s, \pi)$ is positive definite finite or infinite?

My belief is that it is finite.

For references to Section 1 of these lectures see "Recent progress on QUE"

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