Linear Representations and Isospectrality

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Quotients and representations

If a group G acts on a manifold M, Then $L_2(M)$ is a representation of G. For $H \leq G$, functions on the quotient M/H correspond to morphisms from $\mathbf{1}_H$, the trivial representation of H, to $L_2(M)$.

$$\underbrace{L_2(M/H)}_{\text{Functions on } M/H} \cong \underbrace{L_2(M)^{\mathbf{1}_H} = \left\{ f \,\middle|\, \begin{matrix}\forall h \in H \\ hf = f\end{matrix} \right\}}_{H-\text{invariant functions on } M} \cong \underbrace{\mathrm{Hom}_{\mathbb{C}H}(\mathbf{1}_H, L_2(M))}_{H-\text{equivariant morphisms}}$$

Frobenius Reciprocity

Allows us to compare H-morphisms with G-morphisms:

$$\mathrm{Hom}_{\mathbb{C}H}(\mathbf{1}_H, L_2(M)) \cong \mathrm{Hom}_{\mathbb{C}G}\left(\mathrm{Ind}_H^G \mathbf{1}_H, L_2(M)\right)$$

Corollary: Sunada's isospectral construction

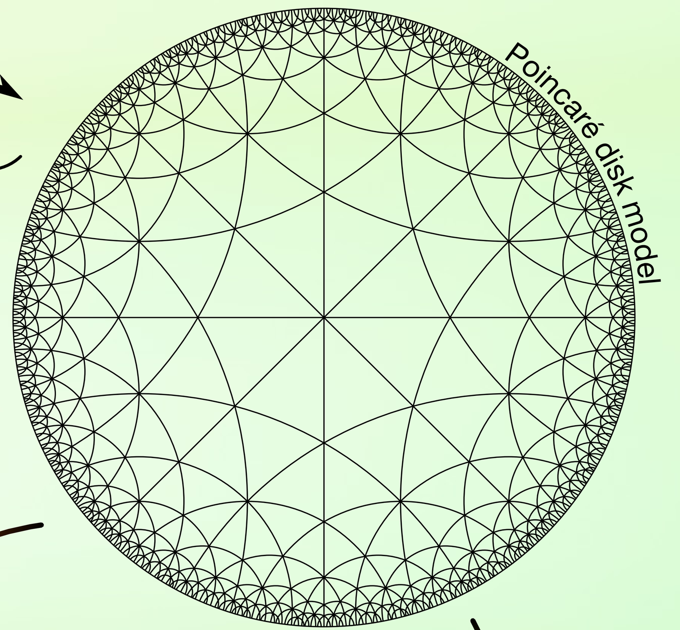
If H,H' are subgroups of G satisfying the Sunada condition

$$\mathrm{Ind}_H^G \mathbf{1}_H \cong \mathrm{Ind}_{H'}^G \mathbf{1}_{H'}$$

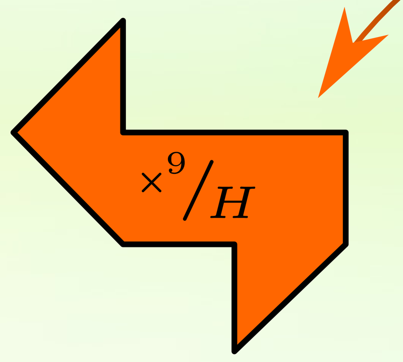
then M/H and M/H' are isospectral.

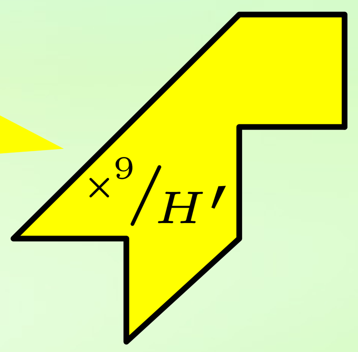
Example: Gordon-Webb-Wolpert Drums

$G = \mathrm{PSL}_3(2)$ acts on x^9 - a quotient of the hyperbolic plane



$H = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ $\quad H' = \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$





Quotients *by* representations

We replace $\mathbf{1}_H$ by **any** representation R of H, and construct an object (denoted M/R) such that there is an isomorphism:

$$\underbrace{L_2(M/R)}_{\text{Functions on the new object}} \cong \mathrm{Hom}_{\mathbb{C}H}(R, L_2(M)) \cong \underbrace{L_2(M)^R = \left\{ f \,\middle|\, \begin{matrix}\forall h \in H \\ hf = \rho_R(h) f\end{matrix} \right\}}_{\text{If } R \text{ is one-dimensional}}$$

Consequence: more isospectrality

If G acts on M, and H,H' are subgroups of G with corresponding representations R,R', then in the same manner (Frobenius Reciprocity)

$$\mathrm{Ind}_H^G R \cong \mathrm{Ind}_{H'}^G R'$$

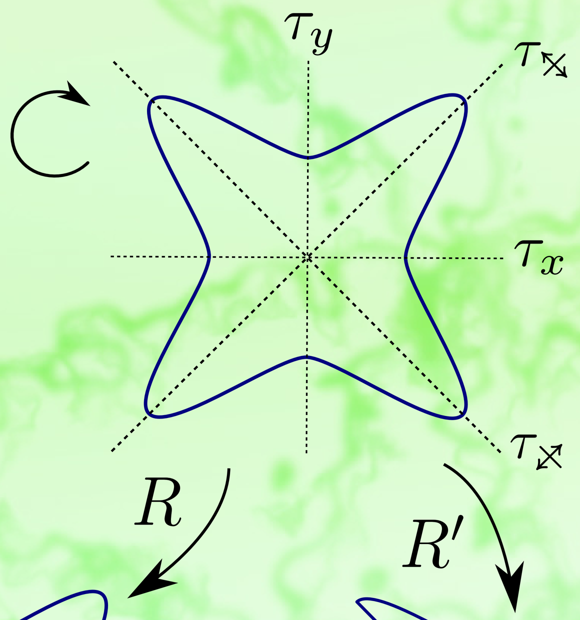
implies that M/R and M/R' are isospectral.

Example: drums with alternating boundary conditions (Jakobson, Levitin et al)

$$G = D_4 = \begin{Bmatrix} id, & \sigma_{90^\circ}, & \sigma_{180^\circ}, & \sigma_{270^\circ}, \\ \tau_x, & \tau_y, & \tau_{\diagdown}, & \tau_{\diagup} \end{Bmatrix}$$

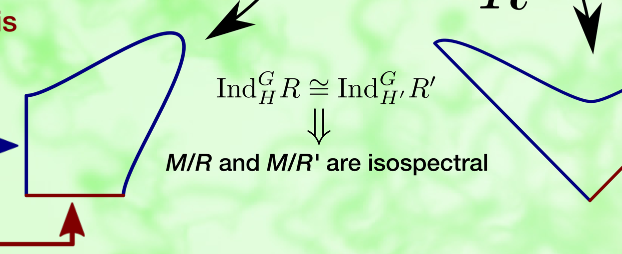
$H = \langle \tau_x, \tau_y \rangle \qquad H' = \langle \tau_{\diagdown}, \tau_{\diagup} \rangle$

$\rho_R : \begin{matrix} \tau_x \mapsto 1 \\ \tau_y \mapsto -1 \end{matrix} \qquad \rho_{R'} : \begin{matrix} \tau_{\diagdown} \mapsto 1 \\ \tau_{\diagup} \mapsto -1 \end{matrix}$

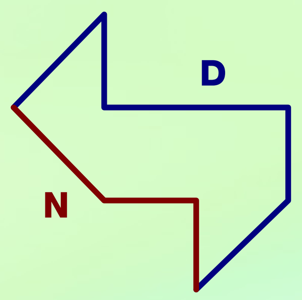


Dirichlet (Dark blue): $\tau_y f = \rho_R(\tau_y) f = -f$ means that f is antisymmetric at the y-axis, hence vanishes there.

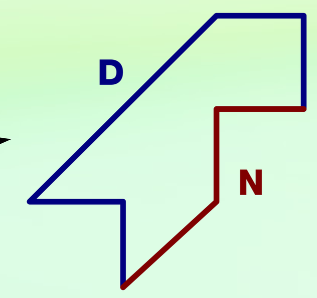
Neumann (Nifty red): $\tau_x f = \rho_R(\tau_x) f = f$ means that f is symmetric at the x-axis, hence its derivative vanishes there.



In the Gordon-Webb-Wolpert construction both H and H' are isomorphic to S_4. Taking their sign representations we obtain:



$$\mathrm{Ind}_H^{\mathrm{PSL}_3(2)} \mathrm{Sgn}_H \cong \mathrm{Ind}_{H'}^{\mathrm{PSL}_3(2)} \mathrm{Sgn}_{H'}$$
$$\Downarrow$$
x^9/Sgn_H is isospectral to $x^9/\mathrm{Sgn}_{H'}$



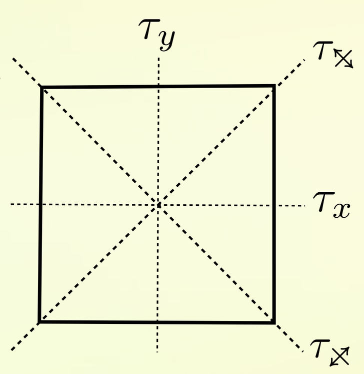
Sums and unions (Chapman)

$L_2(M \cup M') = L_2(M) \oplus L_2(M')$ shows that $M/R \cup M/R' = M/R\oplus R'$.

Taking now four subgroups of D_4

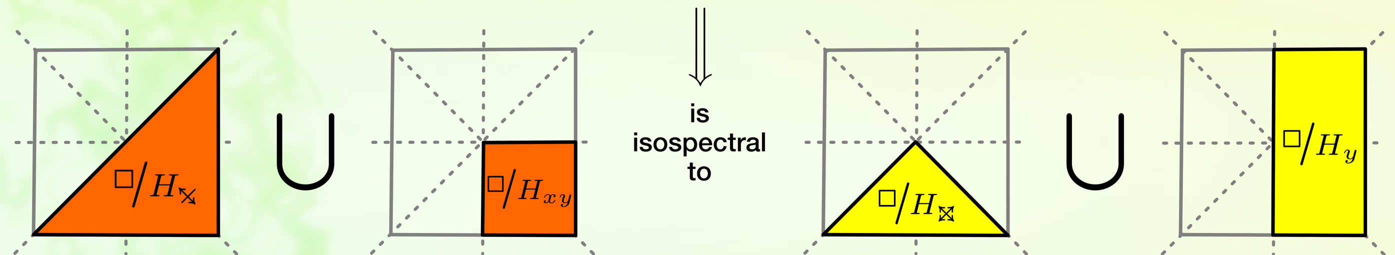
$H_{xy} = \langle \tau_x, \tau_y \rangle \qquad H_{\boxtimes} = \langle \tau_{\diagdown}, \tau_{\diagup} \rangle$

$H_y = \langle \tau_y \rangle \qquad H_{\diagdown} = \langle \tau_{\diagdown} \rangle$



we get:

$$\mathrm{Ind}_{H_{\diagdown}}^{D_4} \mathbf{1}_{H_{\diagdown}} \oplus \mathrm{Ind}_{H_{xy}}^{D_4} \mathbf{1}_{H_{xy}} \cong \mathrm{Ind}_{H_{\boxtimes}}^{D_4} \mathbf{1}_{H_{\boxtimes}} \oplus \mathrm{Ind}_{H_y}^{D_4} \mathbf{1}_{H_y}$$



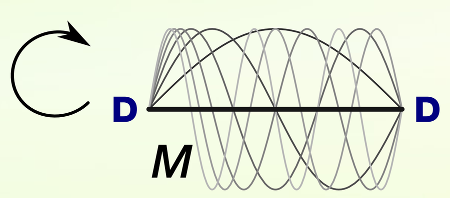
Isospectrality everywhere

(or - things you can do with Z mod 2)

Yes! Even out of this humblest of groups isospectrality can be squeezed!

$G = \mathbb{Z}/2\mathbb{Z} = \{id, \tau\}$

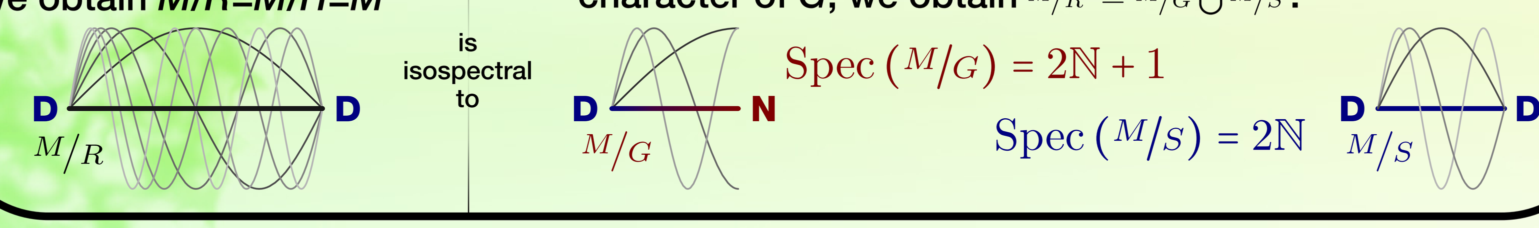
M is a vibrating string with Dirichlet boundary conditions. $\mathrm{Spec}(M) = \mathbb{N}$



Taking $H=\{id\}$ and $R=\mathbf{1}_H$ we obtain $M/R=M/H=M$

is isospectral to

For $H'=G$ and $R' = \mathbb{C}G \cong 1_G \oplus S$, where S is the nontrivial character of G, we obtain $M/R' = M/G \cup M/S$:

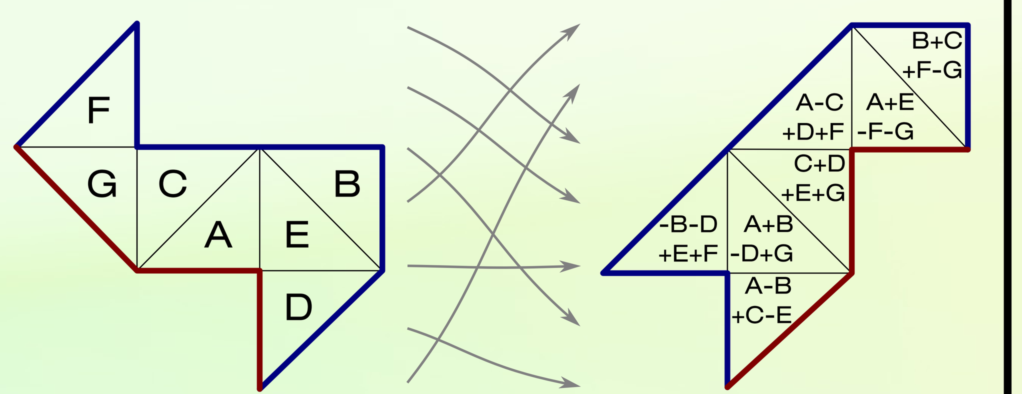


$\mathrm{Spec}(M/G) = 2\mathbb{N}+1$

$\mathrm{Spec}(M/S) = 2\mathbb{N}$

Transplantation

From $\mathrm{Ind}_H^G R \cong \mathrm{Ind}_{H'}^G R'$ a transplantation operator is induced between the quotients, by the composition of isomorphisms:



$$L_2(M/R) \cong \mathrm{Hom}_{\mathbb{C}H}(R, L_2(M)) \cong \mathrm{Hom}_{\mathbb{C}G}\left(\mathrm{Ind}_H^G R, L_2(M)\right) \cong$$
$$\mathrm{Hom}_{\mathbb{C}G}\left(\mathrm{Ind}_{H'}^G R', L_2(M)\right) \cong \mathrm{Hom}_{\mathbb{C}H'}(R', L_2(M)) \cong L_2(M/R')$$

T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of math. **121** 169-186,(1985). M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly, **73** (1966) 1-23. C. Gordon, D. Webb and S. Wolpert, *One cannot hear the shape of a drum*, Bull. Am. Math. Soc. **27** 134-138 (1992). P. Buser, J. Conway, P. Doyle and K.-D. Semmler, *Some Planar Isospectral Domains*, Int. Math. Res. Notices **9** (1994), 391-400. D. Jakobson, M. Levitin, N. Nadirashvili, and I. Polterovich, *Spectral problems with mixed Dirichlet-Neumann boundary conditions: isospectrality and beyond*, J. Comp. and Appl. Math. **194** (2004), 141-155. M. Levitin, L. Parnovski, and I. Polterovich, *Isospectral domains with mixed boundary conditions*, J. Phys. A: Math. Gen., **39** (2005), 2073-2082. S. Chapman *Drums that sound the same*, Amer. Math. Monthly **102** (1995) 124-38.