Parseval frames of exponentially decaying Wannier functions

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$P_H^\perp : H_1 \mapsto H$ - orthogonal projector. (Converse statement clearly also holds.)

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Plane waves $\Leftrightarrow$ δ-**functions relation**: $e^{ix\cdot\xi_0} \overset{FT}{\Leftrightarrow} \delta(\xi - \xi_0)$

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Floquet-Bloch direct integral decomposition:

$$L^2(\mathbb{R}^n) = \int_B^{\oplus} L^2(W)dk = \int_{\mathbb{T}^*}^{\oplus} L^2(W)dz.$$

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Its inversion:

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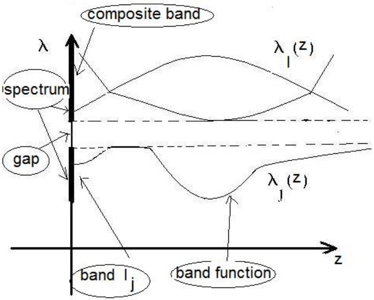

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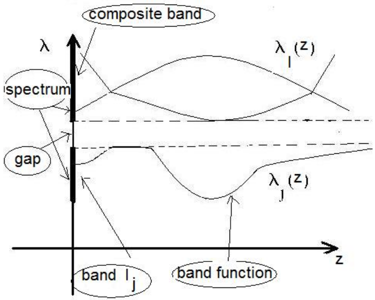

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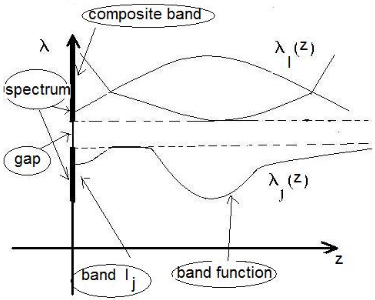

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$$\sigma(L) = \bigcup_{\mathbb{T}} \sigma(L(z))$$

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Triviality holds if $n = 1$ (W. Kohn '59) or if there is **time reversal symmetry** $z \mapsto z^{-1} \Leftrightarrow k \mapsto -k$ and either $m = 1$ (Nenciu '85), or $n \leq 3$ (Panati '07).

Time reversal symmetry occurs if the coefficients of the operator are real (e.g., magnetic fields are excluded).

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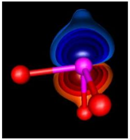

An example of WF in Barium Titanate

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So, what can one do?

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- The number l is the smallest dimension of the fiber of a trivial vector bundle over $\mathbb{T}^*$ that contains an isomorphic copy of Λ_S. In particular, $l \leq 2^n m$.
- $l = m$ iff Λ_S is trivial, in which case there exists an o.-n. basis of exponentially decaying Wannier functions in H_S.

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- Apply to $\{e_j\}$ an analytic projector $P(z)$ onto Λ_S orthogonal over $\mathbb{T}^*$ to get the Wannier functions $\{w_j\}$.

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THANK YOU