Semiclassical limits of eigenfunctions of the Laplacian on $\mathbb{T}^n$

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Setting & notation

- The n-dimensional torus $\mathbb{T}^n$ is defined to be: $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
- Eigenfunctions of Laplacian on $\mathbb{T}^n$ with eigenvalue $\lambda \geq 0$ satisfy:

$$\begin{aligned} \Delta f + \lambda f &= 0, \\ f(x_1, \ldots, x_n) &= f(x_1 \pm 2\pi, \ldots, x_n \pm 2\pi). \end{aligned}$$

- We will work with the complex Fourier expansion,

$$f(x) \sim \sum_{|\xi| = \sqrt{\lambda}} c_\xi e^{i(\xi, x)}.$$

- We will use the following standard notation:

$$f \in L^p \quad \text{if} \quad \int |f|^p d\mu < \infty.$$

$$f \in \ell^p \quad \text{if} \quad \sum_{k=0}^{\infty} |c_k|^p < \infty.$$

Motivating results

- Given an eigenfunction $\varphi(x)$ of the Laplacian on the flat 2-torus, A. Zygmund showed that,

$$\frac{||\varphi||_{L^4}}{||\varphi||_{L^2}} \leq 5^{1/4}.$$

  The bound above is independent of the eigenvalue λ.
- For $n \geq 4$, J. Bourgain showed that on $\mathbb{T}^n$,

$$\sup_{(\Delta+\lambda)\varphi=0} \frac{||\varphi||_{L^p}}{||\varphi||_{L^2}} \ll \lambda^{(n-2)/4-n/2+\varepsilon}$$

  for $p \geq \frac{2(n+1)}{(n-3)}$.

⋆ In fact as $\lambda \to \infty$,

$$\sup_{(\Delta+\lambda)\varphi=0} \frac{||\varphi||_{L^p}}{||\varphi||_{L^2}} = \infty.$$

Lower dimensional case

In [Jak97], D. Jakobson studied several aspects of quantum limits on flat tori. He was interested in the limit as $\lambda_j \to \infty$ of the measure,

$$d\mu_j = |\varphi_j|^2 d\,\mathrm{vol},$$

where φ_j were eigenfunctions of the Laplacian on $\mathbb{T}^n$ with eigenvalue λ_j and (d vol) is the Riemannian volume form.

Theorem (D. Jakobson, N. Nadirashvili, J. Toth)

Let φ_j be an eigenfunction of the Laplacian on $\mathbb{T}^n$. Then, for $2 \leq n \leq 4$, the Fourier series of $|g| := |\varphi_j|^2$ has a uniform ℓ^n norm, where the bound is independent of the eigenvalue λ_j.

The theorem stated above implies a statement about limits of eigenfunctions on $\mathbb{T}^{n+2}$. That is, *quantum limits* have a uniform ℓ^n norm on $\mathbb{T}^{n+2}$.

Higher dimensional case

It was conjectured in [Jak97] that the previous result holds on $\mathbb{T}^n$ for all $n \geq 4$.

Theorem (T. Aïssiou)

For any $n \geq 5$ there exists a constant $C(n) < \infty$, independant of the eigenvalue λ_j, such that for every L^2-normalized eigenfunction of the Laplacian on $\mathbb{T}^n$, the Fourier series of $|g| := |\varphi_j|^2$ has a uniform ℓ^n norm. That is,

$$||\widehat{g}||_{\ell^n} \leq C(n) ||\varphi_j||_{L^2}^2 .$$

The proof of the 3 dimensional case is given in [Jak97], the 4 dimensional case in [JNT01] and the general n dimensional case in [Ais09].

Remarks

- The bound $C(n)$ depends on the dimension n only. As $n \to \infty$, $C(n) \to 2$. In fact,

$$C(n) = \left(2^{2-n} + \left(\frac{5n}{4} - 4\right) 2^n + 5\right)^{1/n}.$$

- Combining the last two theorems and the result about quantum limits proved in [Jak97], we can show that quantum limits on $\mathbb{T}^{n+2}$ have a uniform ℓ^n norm for any $n \geq 3$.
- The proof is done by induction on the dimension and uses a geometric lemma that bounds the number of codimension-one simplices on the n dimensional sphere $\mathbf{S}(\lambda_j)$ of radius λ_j.

Geometric lemma and key remarks.

Lemma (Geometric Lemma)

Given n points $\{\xi_i\}_{i=1}^n$ *on* $\mathbf{S}^{n-1}(\lambda_j) \cap \mathbb{Z}^n$, *no two of which are diametrically opposite, that form codimension-one simplex, assume that there exists* $\tau \in \mathbb{Z}^n$ *and another n points* $\{\eta_i\}_{i=1}^n$ *on* $\mathbf{S}^{n-1}(\lambda_j) \cap \mathbb{Z}^n$ *such that,*

$$\xi_i - \eta_i = \tau, \qquad \forall 1 \leq i \leq n. \tag{1}$$

Then, there can be at most 2^{n-1} *such different vectors* τ *satisfying* (1).

- Given $m > n$ points on $\mathbf{S}^{n-1}(\lambda_j) \cap \mathbb{Z}^n$, we will still have the same bound, 2^{n-1} on the number of possible τ's.
- The bound we obtained in the lemma is independent of the eigenvalue λ_j. This fact is crucial in the proof of the main Theorem.

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