Permutation Patterns in Algebraic Geometry

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We start with some definitions.

Permutations and patterns

A **permutation** in $\mathfrak{S}_n$ is a bijection $\pi \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$.
We will use one-line notation for permutations, for example, $\pi = 32415$ is the permutation in $\mathfrak{S}_5$ that sends

$$1 \mapsto 3$$
$$2 \mapsto 2$$
$$3 \mapsto 4$$
$$4 \mapsto 1$$
$$5 \mapsto 5.$$

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$$\begin{aligned} 1 &\mapsto 3 \\ 2 &\mapsto 2 \\ 3 &\mapsto 4 \\ 4 &\mapsto 1 \\ 5 &\mapsto 5. \end{aligned}$$

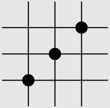
Patterns are also permutations but we are interested in how they occur in other permutations . . .

Patterns inside permutations

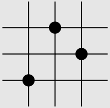
Given a pattern p we say that it **occurs** in a permutation π if π contains a subsequence that is order-equivalent to p. If p does not occur in π we say that π **avoids** the pattern p. Let $\mathfrak{S}_n(p)$ denote the set of permutations in $\mathfrak{S}_n$ that avoid the pattern p.

Example

The permutation $\pi = 32415$ has two occurrences of the pattern

$123 =$  : **3**2**4**1**5**, 3**24**1**5**

It avoids the pattern

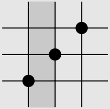
$132 =$  .

Vincular patterns

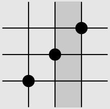
Babson and Steingrímsson (2000) defined **generalized patterns**, or **vincular patterns**, where conditions are placed on the locations of the occurrence.

Example

The permutation $\pi = 32415$ has one occurrence of the pattern

$\underline{12}3 =$  : 3**2415**

It avoids the pattern

$1\underline{23} =$  .

Motivation for vincular patterns

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- They simplify descriptions given in terms of more complicated patterns – we'll see this later when we look at factorial Schubert varieties.
- Many interesting sequences of integers come up when we count the permutations avoiding a pattern p. For example if p is any classical pattern of length 3 then

$$|\mathfrak{S}_n(p)| = n\text{-th Catalan number} = \frac{1}{n+1}\binom{2n}{n}.$$

However Claesson showed in 2001 that

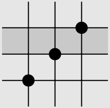
$$|\mathfrak{S}_n(1\underline{23})| = n\text{-th Bell number}.$$

Bivincular patterns

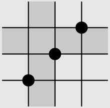
Bousquet-Mélou, Claesson, Dukes, and Kitaev (2010) defined **bivincular patterns** as vincular patterns with extra restrictions on the values in an occurrence.

Example

The permutation $\pi = 32415$ has one occurrence of the pattern

$$\begin{matrix} 1\overline{23} \\ 123 \end{matrix} =$$  : 3**2**4**1**5...

This is not an occurrence of $\begin{matrix} \overline{12}3 \\ 123 \end{matrix}$. But it is an occurrence of

$$\begin{matrix} 1\overline{23} \\ \underline{12}3 \end{matrix} =$$  : 3**2**4**1**5

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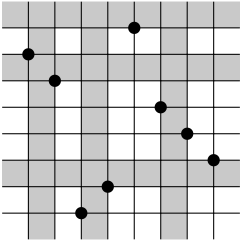
- More interesting sequences of integers.
- New Wilf-equivalence: For example the patterns

$$\begin{matrix} 1\overline{23}45\overline{67}8\rceil \\ \underline{76}\,\underline{12}\,8\,\underline{54}\,3 \end{matrix} =$$



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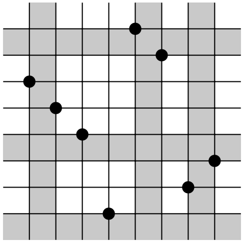
$$\begin{array}{c} \ulcorner 12\overline{34}56\overline{78} \\ \underline{65}41\underline{87}\,\underline{23} \end{array} =$$ 

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(Complete) flags

We will only consider complete flags in $\mathbb{C}^m$ so we will simply refer to them as **flags**. A flag is a sequence of vector-subspaces of $\mathbb{C}^m$

$$E_\bullet = (E_1 \subset E_2 \subset \cdots \subset E_m = \mathbb{C}^m),$$

with the property that $\dim E_i = i$. The set of all such flags is called the (**complete**) **flag manifold**, and denoted by $F\ell(\mathbb{C}^m)$.

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Schubert cells in $F\ell(\mathbb{C}^m)$

If we choose a basis $f_1, f_2, \ldots, f_m$, for $\mathbb{C}^m$ then we can fix a **reference flag**

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for $1 \leq p, q \leq m$.

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$$\dim(E_p \cap F_q) = \#\{i \leq p \,|\, \pi(i) \leq q\},$$

for $1 \leq p, q \leq m$. Let's look at an example.

A Schubert cell in $F\ell(\mathbb{C}^3)$

Let $\pi = 231$.The conditions for the Schubert cell X°_{231}

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$q=2$	1	1	2	$E_1 \subset F_2$, $E_2 \cap F_2 = E_1$
$q=3$	1	2	3	

Schubert varieties in $F\ell(\mathbb{C}^m)$

Given a Schubert cell X_π° we define the **Schubert variety** as the closure

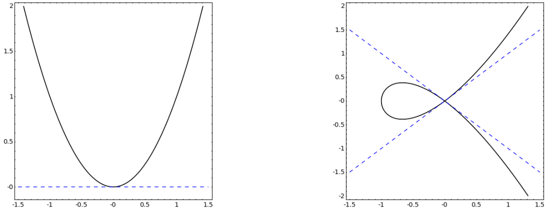
$$X_\pi = \overline{X_\pi^\circ},$$

in the Zariski topology.

We will now show how pattern avoidance can be used to describe geometric properties of Schubert varieties.

Smooth, factorial and Gorenstein varieties

Pictorial definition of smoothness: the tangent space at every point has the right dimension.



(a) $y - x^2 = 0$.

(b) $y^2 - x^2 - x^3 = 0$.

Figure: Compare the single tangent direction in subfigure 1(a) with the two tangent directions in subfigure 1(b).

Smooth, factorial and Gorenstein varieties

Algebraic definitions: a variety:

X is	if
smooth	
factorial	
Gorenstein	

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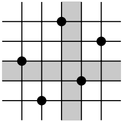
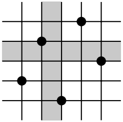
- The first condition of factoriality, avoiding $2\underline{14}3$, is weakened to

avoiding $\begin{matrix}1\,2\,\overline{3\,4}\,5\\ 3\,\underline{1\,5}\,2\,4\end{matrix}$ =  and $\begin{matrix}1\,\overline{2\,3}\,4\,5\\ 2\,4\,\underline{1\,5}\,3\end{matrix}$ = .

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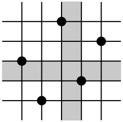
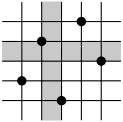
  avoiding $\begin{matrix}12\overline{34}5\\ 3\underline{15}24\end{matrix}$ =  and $\begin{matrix}1\overline{23}45\\ 24\underline{15}3\end{matrix}$ = .

- The second condition of factoriality, avoiding 1324, is weakened to the avoidance of two infinite families of bivincular patterns, which we now describe.

The associated partition of a permutation

Here we will only consider permutations with a unique descent, as this allows us to avoid a minor technical detail.

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Here we will only consider permutations with a unique descent, as this allows us to avoid a minor technical detail.
Given such a permutation π, with a descent at d, we construct its **associated partition** $\lambda(\pi)$ as the partition inside a bounding box with dimensions $d \times (n - d)$, whose lower border is the lattice path that starts at the lower left corner of the box and whose i-th step is vertical if i is weakly to the left of the position d, and horizontal otherwise.

Example

The permutation

$$\pi = 134892567|10$$

has a unique descent at $d = 5$.

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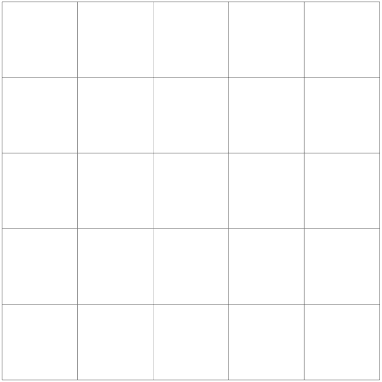

Figure: A bounding box with dimensions 5×5.

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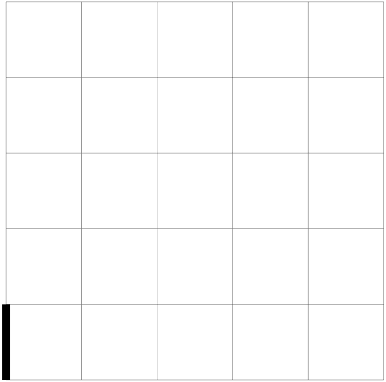

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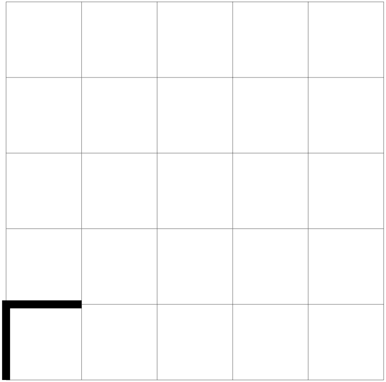

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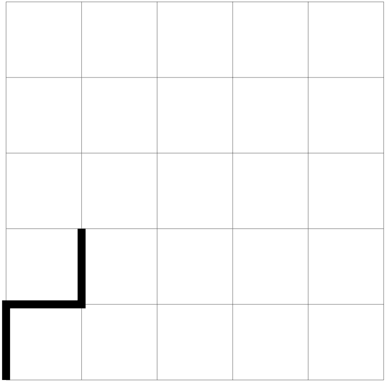

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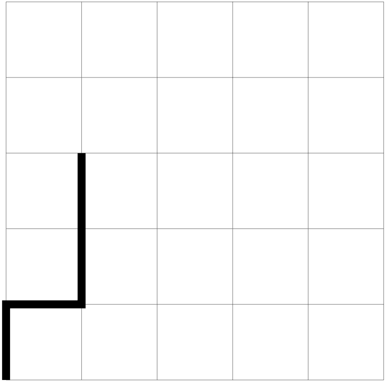

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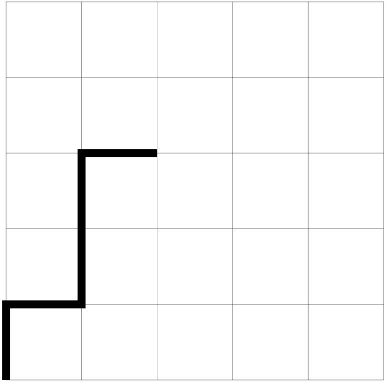

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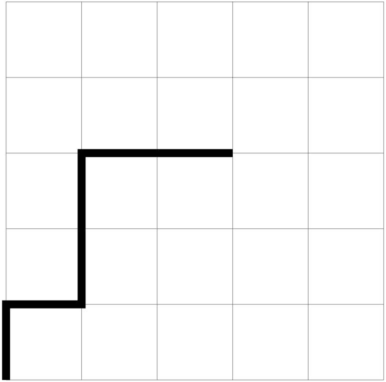

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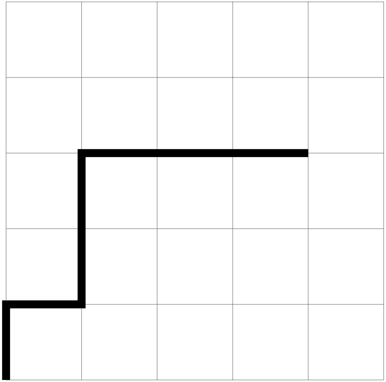

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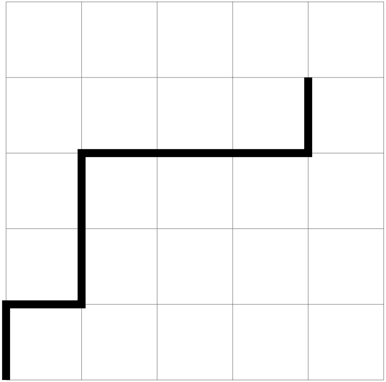

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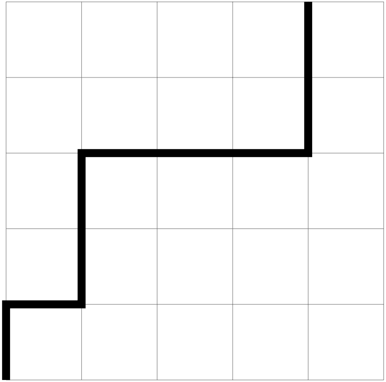

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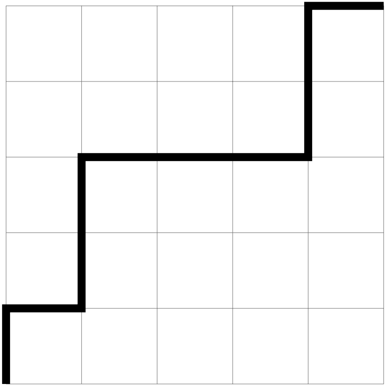

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Inner corners of the partition

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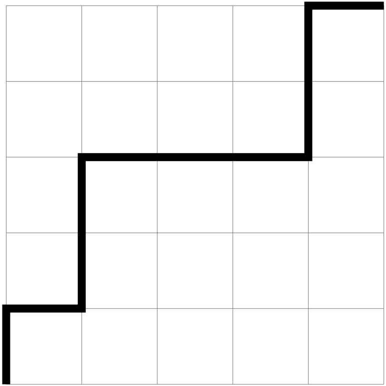

Figure: Inner corners of $\pi = 13489 \downarrow 2567|10$.

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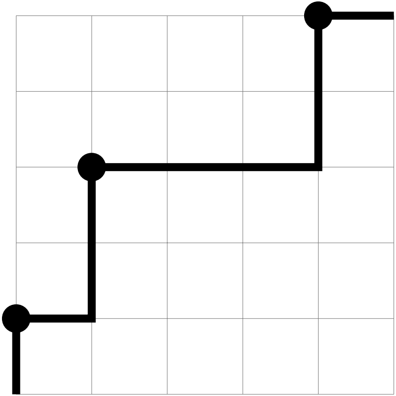

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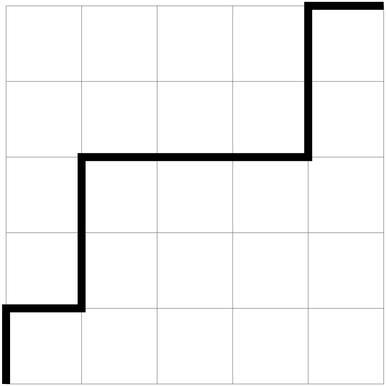

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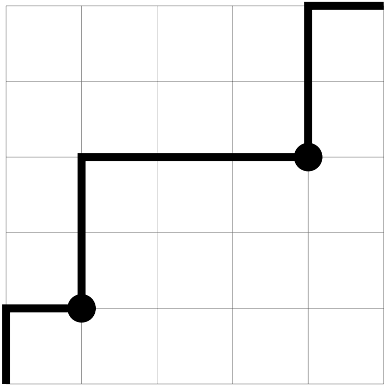


Figure: Outer corners of $\pi = 13489 \downarrow 2567|10$.

Depth and width of outer corners

We see that all the inner corners lie on the same diagonal if and only each outer corner has the same **depth** and **width**.

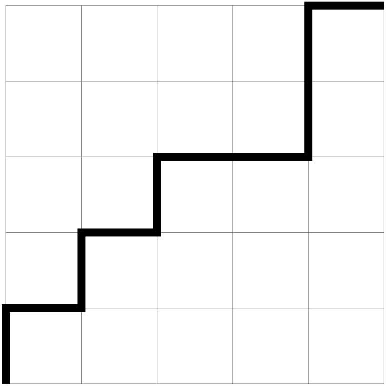
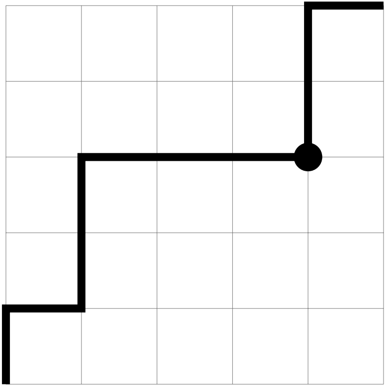


Figure: $\pi = 13589 \downarrow 2467|10$.

Detecting too wide outer corners

Let's go back to the permutation $\pi = 13489 \downarrow 2567|10$, and consider the outer corner that is too wide



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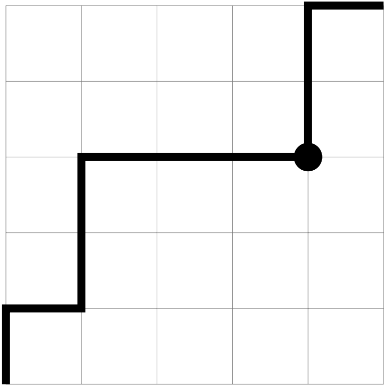
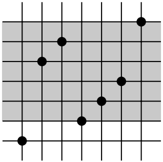


This outer corner comes from the subsequence $134\mathbf{89} \downarrow 2\mathbf{567}|\mathbf{10}$.

Detecting too wide outer corners cont.

The shape of this outer corner can be detected with the bivincular pattern

$$\begin{matrix} 1\overline{234567} \\ 1562347 \end{matrix} =$$



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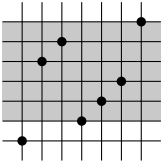
$$\genfrac{}{}{0pt}{}{1\overline{234567}}{1562347} =$$  .

In general, we can detect too wide outer corners with the patterns

$$\genfrac{}{}{0pt}{}{1\overline{2345}}{14235},\ \genfrac{}{}{0pt}{}{1\overline{234567}}{1562347},\ \genfrac{}{}{0pt}{}{1\overline{23456789}}{167823459}, \ldots,\ \genfrac{}{}{0pt}{}{1\overline{2\cdots\cdots\cdots k}}{1\,\ell{+}1\cdots 2\cdots \ell k}, \cdots$$

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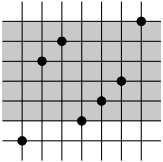
$$\genfrac{}{}{0pt}{}{1\overline{234567}}{1562347} =$$  .

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and too deep outer corners with the patterns

$$\genfrac{}{}{0pt}{}{\overline{1234}5}{13425}, \genfrac{}{}{0pt}{}{\overline{123456}7}{1456237}, \genfrac{}{}{0pt}{}{\overline{12345678}9}{156782349}, \dots, \genfrac{}{}{0pt}{}{\overline{12\cdots\cdots\cdots}k}{1\,\ell{+}1\cdots 2\cdots \ell\, k}, \cdots$$

Summary

The Schubert variety

X_π is	if
smooth	π avoids 2143 and 1324
factorial	π avoids $2\underline{14}3$ and 1324
Gorenstein	π avoids $\begin{matrix}12\overline{34}5\\3\underline{15}24\end{matrix}$ and $\begin{matrix}1\overline{23}45\\24\underline{15}3\end{matrix}$, ...

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... and the two infinite corner families — remember that this is modulo a technical detail I have omitted.

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- The description is in terms of patterns only and one doesn't need to construct the associated partition.
- It is very easy to see on the pattern level that smooth implies factorial implies Gorenstein.

We end with some open problems.

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- Where do the **mesh patterns** patterns fit into this story?

The end! Questions?