231-Avoiding Permutations and the Schensted Correspondence

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- Each block is obviously 231-avoiding.
- Each block has the largest element at the left.
- If σ, τ are 231-avoiding, then so are $\sigma \oplus \tau$ and $1 \ominus \sigma$.
- Therefore, a permutation σ is 231-avoiding iff it can be generated by these two operations and 1.

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- Length of bottom row $=$ length of longest increasing subsequence
- Length of left column $=$ length of longest decreasing subsequence

Effect of $\oplus$ and $\ominus$ on RSK

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- $312 \ominus 3142 = 7563142$:
- $RSK(7563142) = \left(\begin{array}{cc} 7 & \\ 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{array}, \begin{array}{cc} 5 & \\ 4 & 7 \\ 2 & 6 \\ 1 & 3 \end{array} \right)$

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- Let $T_{k,n} = \{\sigma \in S_n(231) : \forall i > k, \lambda_i(\sigma) = 0\}$.
- Finally, define the generating function $f_k(z)$, short for $f_k(\mathbf{x}, q, z)$, by:

$$f_k(z) = \sum_{n \geq 0} \sum_{\sigma \in T_{k,n}} w(\sigma) q^{\mathrm{inv}(\sigma)} z^n.$$

A Recurrence for f_k

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- Thus:

$$f_k(z) = \left(1 - x_1 z - \sum_{i=2}^{k} x_i z(f_{i-1}(zq) - f_{i-2}(zq))\right)^{-1}$$

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- Note that if you replace x_i with $x_i(y_i/y_{i-1})$, with $y_0 = 1$, then y_i counts the number of columns with i cells.

- A *peak* in a permutation σ is a segment $\sigma_i \sigma_{i+1} \sigma_{i+2}$ such that $\sigma_i < \sigma_{i+1} > \sigma_{i+2}$.

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- Let $\mathrm{peak}(\sigma)$ denote the number of peaks in σ.
- Finally, define the generating function $F(z)$ by:

$$F(z) = \sum_{n \geq 0} \sum_{\sigma \in S_n(231)} p^{\mathrm{peak}(\sigma)} q^{\mathrm{inv}(\sigma)} z^n.$$

Solving for $F(z)$

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- $$F(z) = 1 - \frac{1}{p} + \frac{1}{p}\cfrac{1+(p-1)z}{1 - z(\cfrac{1+(p-1)zq}{1 - zq(\cfrac{1+(p-1)zq^2}{1 - zq^2(\ddots)})})}$$

231-avoiding Permutations by Peaks and Schensted Information

Define $F_k(z)$ by:

$$F_k(z) = \sum_{n\geq 0} \sum_{\sigma \in T_{k,n}} p^{\mathrm{peak}(\sigma)} x^{\lambda_1(\sigma)} q^{\mathrm{inv}(\sigma)} z^n.$$

Then:

$$F_k(z)=1-\frac{1}{p}+\frac{1}{p}\cfrac{1+(p-1)zx}{1-z(x-1+\cfrac{1+(p-1)zqx}{1-zq(x-1+\cfrac{1+(p-1)zq^2x}{1-zq^2(x-1+\ddots\cfrac{1+(p-1)zq^{k-1}x}{1-zq^{k-1}x})})})}$$

Thanks for listening!