Grid Pattern Classes

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Permutation Patterns, Dartmouth College, 11 Aug 2010

Gridding a permutation

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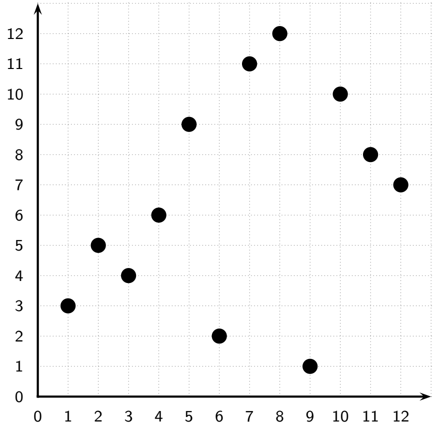

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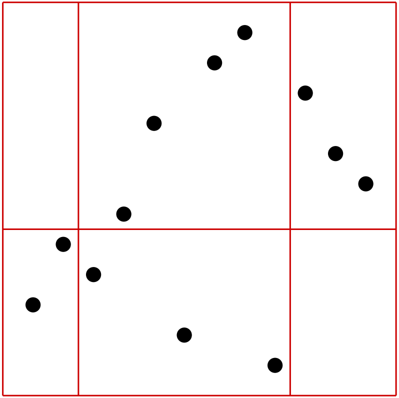

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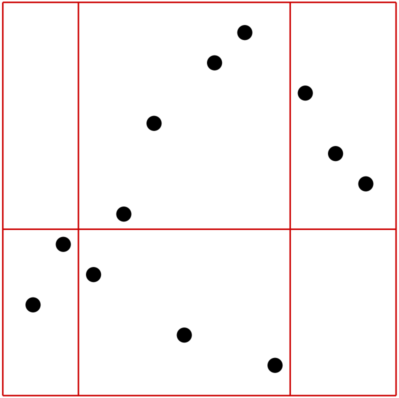

$$M = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

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- Permutation π admits an M-gridding if the xy-plane with the graph Γ of π plotted in it can be partitioned into an axis parallel rectangular grid C_{ij} ($i \in [p]$, $j \in [q]$) such that

$$\Gamma \cap C_{ij} \text{ is } \begin{cases} \text{increasing,} & \text{if } m_{ij} = 1 \\ \text{decreasing,} & \text{if } m_{ij} = -1 \\ \emptyset, & \text{if } m_{ij} = 0. \end{cases}$$

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- The (pattern class) of all π that admit an M-gridding is called a (monotone) grid class, and denoted Grid(M).

Outline & Credits

- Early occurrences of grid classes in literature.
- New project: Basic properties of grid classes.
- Interplay of permutation, geometrical and language-theoretical methods.
- Future directions.

Collaborators: Michael Albert (Otago), Mike Atkinson (Otago), Mathilde Bouvel (Paris/Bordeaux), Robert Brignall (Bristol/OU), Vincent Vatter (Dartmouth/Florida), NR.

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Matrix: $\begin{pmatrix} -1 & 1 \end{pmatrix}$

Description: juxtapositions of a decreasing and an increasing sequence.

Basis: 231, 132.

Enumeration: 2^{n-1}.

Further examples (1)

Example

Atkinson (1999) proves

$$\operatorname{Av}(321, 2143) = \operatorname{Grid}\begin{pmatrix} 1 & 1 \end{pmatrix} \cup \operatorname{Grid}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and derives as a consequence the enumeration for $\operatorname{Av}(321, 2143)$:

$$2^{n+1} - 2n - 1 - \binom{n+1}{3}.$$

Further examples (2)

Example

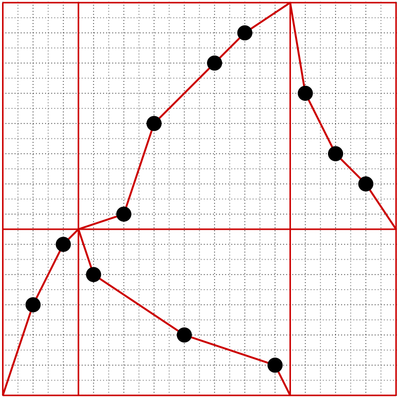
Murphy (2003) expresses several classes with two basis permutations of lengths 3 and 4 as grid classes. For instance

$$\mathrm{Av}(132, 4312) = \mathrm{Grid}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Other occurrences in literature

- Atkinson (1998) – skew merge permutations.
- Profile classes (Atkinson (1999)) – permutation matrices.
- W-classes (Atkinson, Murphy, NR (2002)) – $1 \times q$ matrices.
- Murphy, Vatter (2002) – PWO.
- Waton (2007) – atomicity, Grid $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
- Huczynska, Vatter (2006) – sparse matrices; decidability for subclasses of grid classes.
- Vatter (submitted) – small growth rates.
- Brendan 25,081: Blend includes art arranged in rectangular array.

Playing with the picture



Graph $\Gamma(M)$: definition

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Definition

If $\Gamma(M)$ is a forest we say that $\mathrm{Grid}(M)$ is a forest grid class.

Graph $\Gamma(M)$: examples

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$$\begin{array}{|c|c|c|}\hline & 1 & -1 \\ \hline 1 & -1 & \\ \hline\end{array}$$

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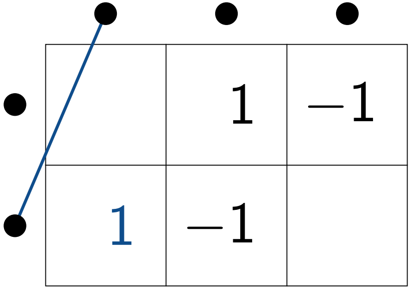

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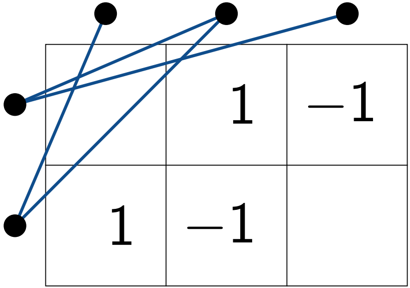

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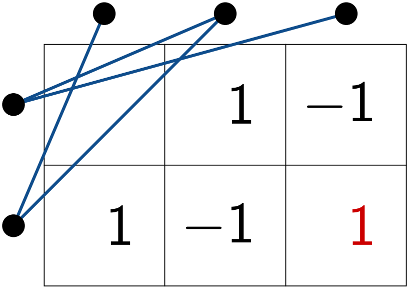

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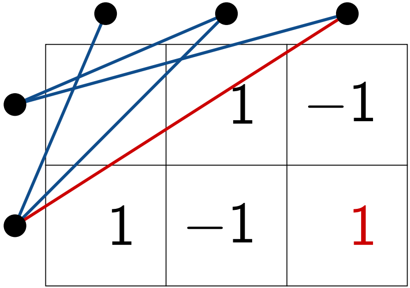

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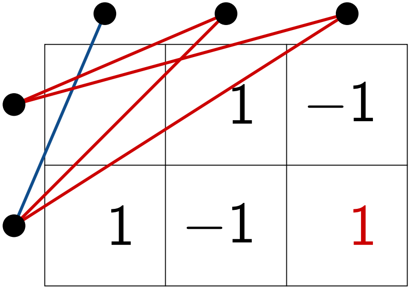

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A finitely based class is PWO iff it has $\leq \aleph_0$ *subclasses iff all its subclasses are finitely based.*

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Theorem (Murphy, Vatter (2002))

A grid class is PWO iff it is a forest grid class.

Important things

He took the biscuits carefully out of the packet and laid them face upward on the grass, in order as he felt of edibility. They were the same as always, a Ginger, an Osborne, a Digestive, a Petit Beurre and one anonymous. He always ate the first-named last, because he liked it the best, and the anonymous first, because he thought it very likely the least palatable. The order in which he ate the remaining three was indifferent to him and varied irregularly from day to day. On his knees now before the 5 it struck him for the first time that this reduced to a paltry six the number of ways in which he could make his meal. (...) Even if he conquered his prejudice against the anonymous, still there would be only twenty-four ways in which the biscuits could be eaten. But were he to take the final step and overcome his infatuation with the ginger, then the assortment would spring to life before him, dancing the radiant measure of its total permutability, edible in a hundred and twenty ways!

(*S. Beckett, Murphy*)

Geometric classes

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- Waton (et al.): permutations from parallel lines; circle permutations; convex permutations; etc.

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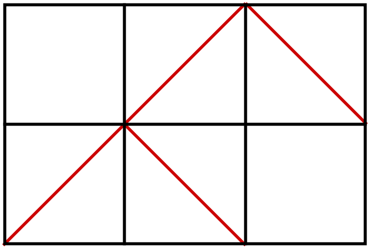

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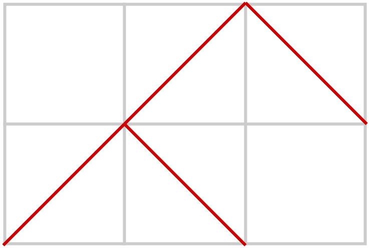

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- $\mathrm{GGrid}(M) \subseteq \mathrm{Grid}(M)$.
- $\mathrm{GGrid}(M) = \mathrm{Grid}(M)$ *if* $\Gamma(M)$ *is a forest.*

Finite basis (in words)

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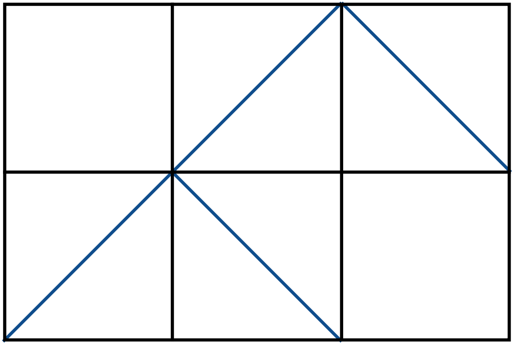

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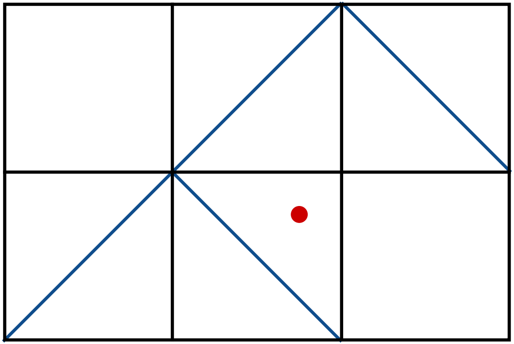

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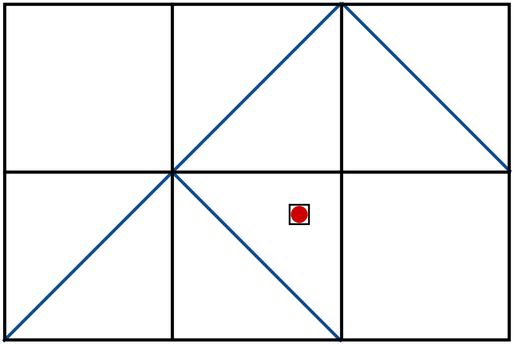

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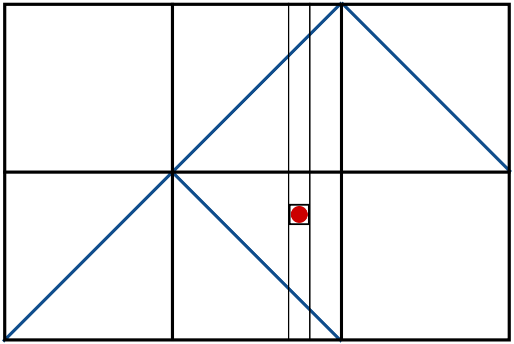

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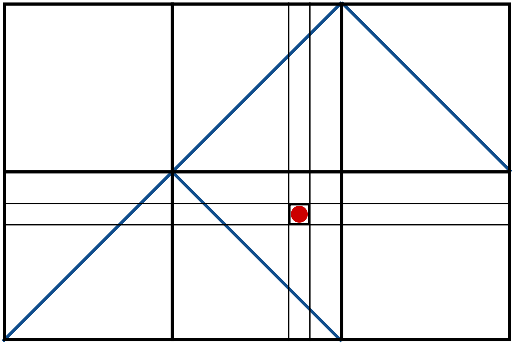

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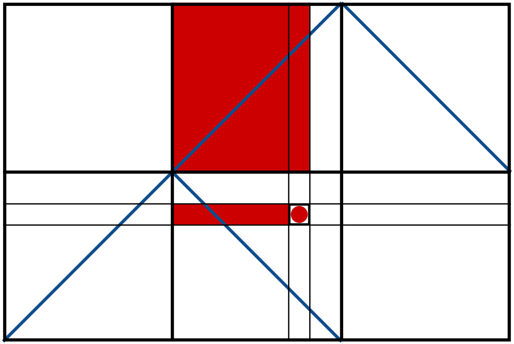

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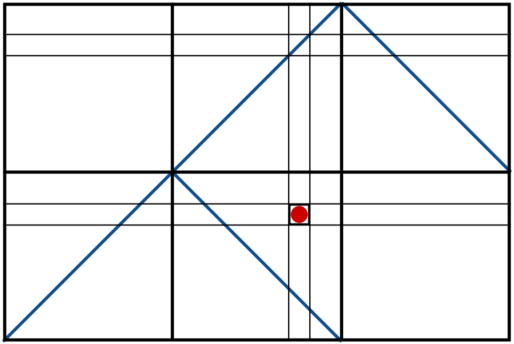

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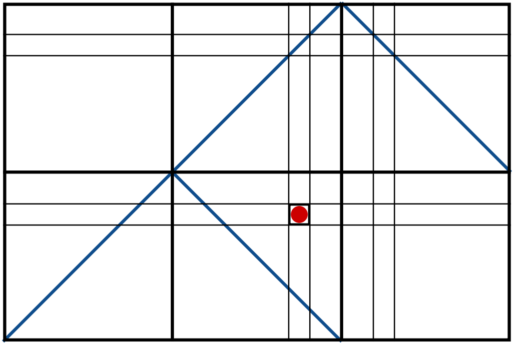

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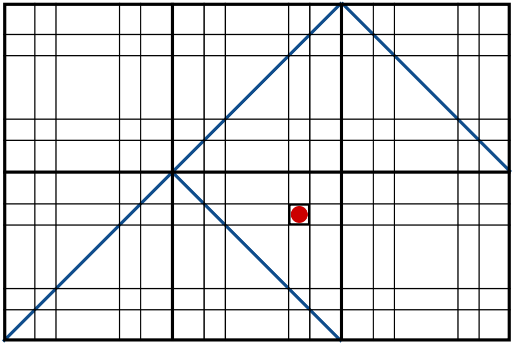

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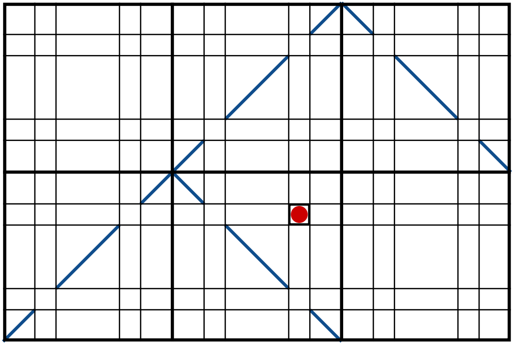

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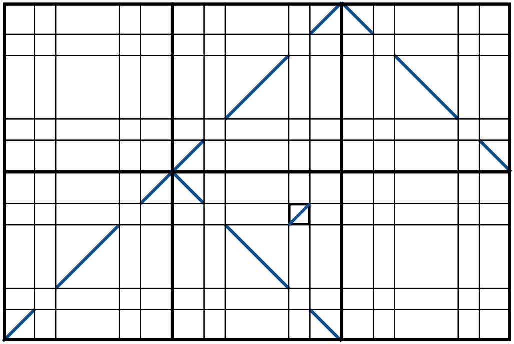

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0	0	0	0	0	0	0	1	0	0	0	0	−1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	−1
0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	$\boxed{1}$	0	0	0	0	0	0
0	0	1	0	0	0	0	−1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	1	0	0	0	0	0	0	0	0	−1
0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	$\boxed{1}$	0	0	0	0	0	0
0	0	1	0	0	0	0	−1	0	0	0	0	0	0	0
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Corollary

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- Every closed subset of A^* is a finite union of sets of the form

$$a_1 A_1^* a_2 A_2^* \dots a_k A_k^* a_{k+1}$$

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  with $a_i \in A$, $A_j \subseteq A$.
- Every closed subset of A^* has a rational generating function.

Encoding gridded permutations as words

Decoding words

- The 'inverse' process.
- $M = (m_{ij})_{p \times q}$, $\Gamma(M)$ a forest.
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- $w_1 \preceq w_2 \Rightarrow \phi(w_1) \preceq \phi(w_2)$.

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Remark

Here 'forest grid class' stands for a small generalisation: one point cells are also allowed.

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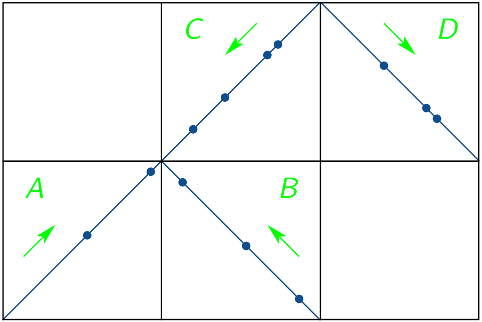

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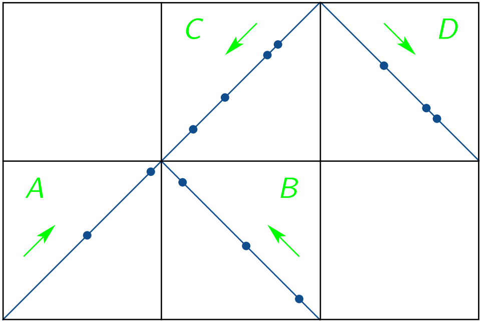

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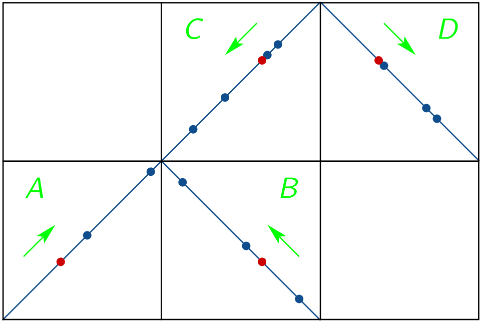

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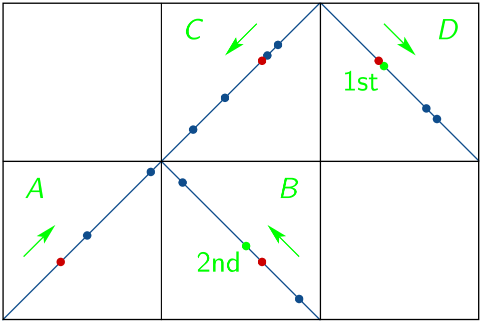

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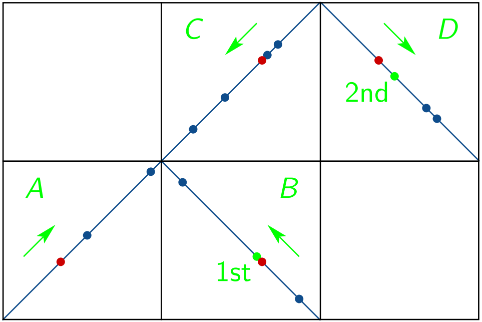

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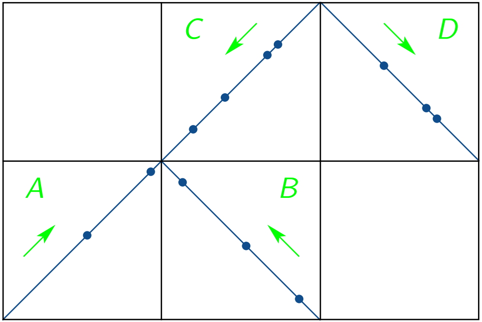

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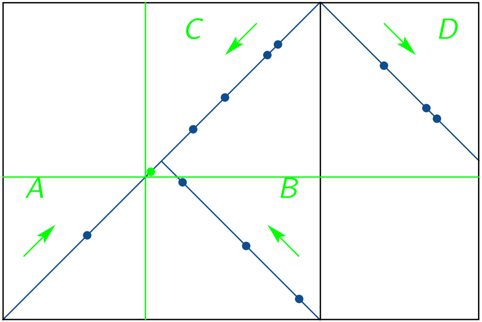

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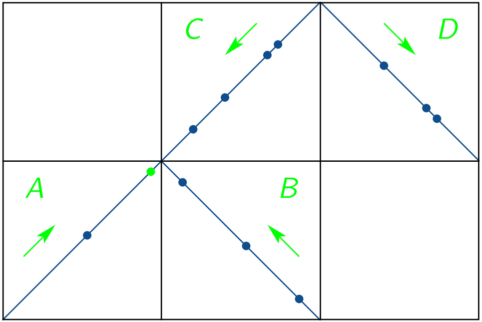

Resolving ambiguities

Independent cells:

- $a_{ij}a_{st} = a_{st}a_{ij}$ where $i \neq s$, $j \neq t$.
- Trace monoids, regular sets of representatives with uniqueness.

Multiple griddings:

- There are only finitely many different gridline movements, relative to the geometric representation.
- An argument similar to basis and subclasses.

Corollary

Every forest grid class has a rational generating function.

Summary of results

For a forest grid class Grid(M) the following hold:

- Grid($\mathcal{M}$) is finitely based.
- Grid($\mathcal{M}$) is partially well ordered.
- Every subclass of Grid($\mathcal{M}$) is a finite union of (slightly generalised) forest grid classes.
- Grid($\mathcal{M}$) and each of its subclasses have rational generating functions.
- The basis and the generating function for Grid($\mathcal{M}$) can be effectively computed from $\mathcal{M}$.
- The sets of $\oplus$-indecomposable, $\ominus$-indecomposable and simple permutations in Grid(M) all have rational generating functions.

Out of the woods?

Conjecture

Every grid class is finitely based.

Conjecture

Every grid class has an algebraic generating function.

Question

Are the basis and the generating function for a grid class algorithmically computable from the gridding matrix?

Question

Is there an algorithm which decides whether a finitely based pattern class (given by its basis) is a grid class?

Evidence

- Waton (2007): Grid $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is finitely based.
- Stankova (1994), Atkinson (1998): Grid $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ has basis $\{2143, 3412\}$ and generating function

$$\frac{1-3x}{(1-2x)\sqrt{1-4x}}.$$

Further directions

Generalised grid classes: Allow cells in the matrix to contain arbitrary pattern classes.

V. Vatter, Small permutation classes, submitted.

R. Brignall, Grid classes and partial well order, submitted.

Thank you!