Consecutive Patterns: From Permutations to Column-Convex Polyominoes and Back

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- Let $\mathcal{PW}$ = set of consecutive pattern problems on words.

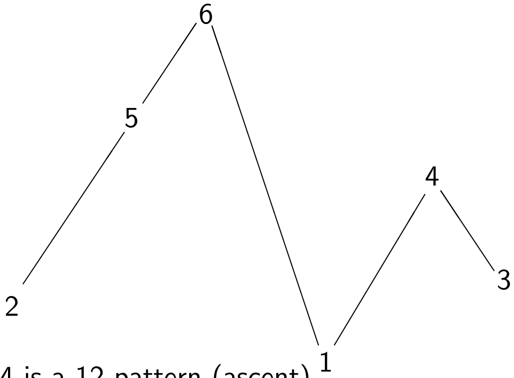
Main Result

$$\mathcal{PS} \subset \mathcal{PC} \subset \mathcal{PCCP} \subset \mathcal{PW}.$$

Remark: Our perspective allows powerful methods from the contexts of compositions, column-convex polyominoes, and of words to be applied directly to the enumeration of permutations by consecutive patterns.

Consecutive Patterns in Permutations

$\sigma = 2\,5\,6\,1\,4\,3 =$



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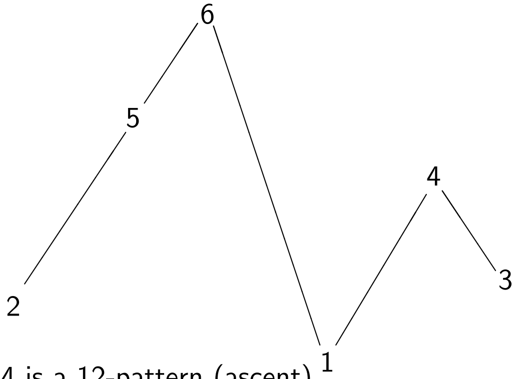

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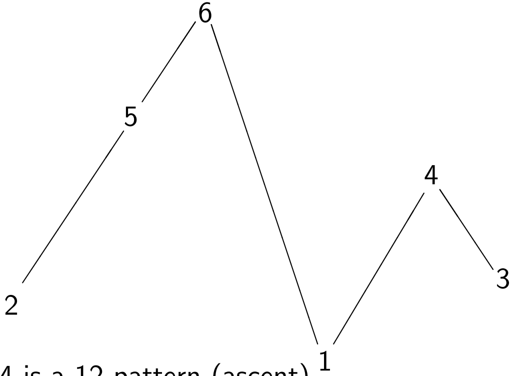

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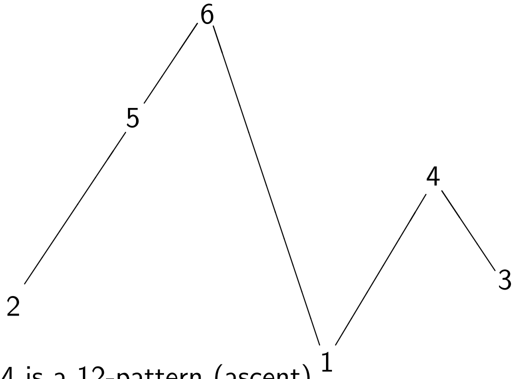

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Definition: For a pattern set $P \subseteq \bigcup_{m\geq 1} S_m$,

- $P(\sigma) =$ the total number of times elements of P occur in σ and
- $P_{no}(\sigma) =$ the maximum number of non-overlapping times elements of P occur in σ.

Consecutive Patterns in Compositions

Notation: $K_n = \{w = w_1 w_2 \ldots w_n : w_1, w_2, \ldots, w_n \text{ are positive integers}\}$

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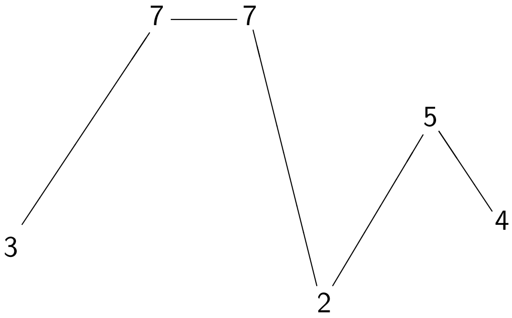
$w = 3\,7\,7\,2\,5\,4 =$



Examples: $w_2 w_3 = 7\,7$ is a level

$w_2 w_3 w_4 = 7\,7\,2$ is a peak $(w_2 \leq w_3 > w_4)$

Inverse of Fedou's Insertion-Shift Bijection

Definitions: For $\sigma \in S_n$, set $\mathrm{inv}_i\sigma = |\{k : i < k \leq n, \sigma_i > \sigma_k\}|$.
Put $\Lambda_n = \{w \in K_n : w_1 \leq w_2 \leq ... \leq w_n\}$.

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The inverse of Fedou's insertion-shift bijection $\nabla_n : S_n \times \Lambda_n \to K_n$ is given by the rule $\nabla_n(\sigma, \lambda) = w$ where $w_i = \mathrm{inv}_i\sigma + \lambda_{\sigma_i}$.

$$\nabla_6(2\,5\,6\,1\,4\,3,\ 2\,2\,4\,4\,4\,4) = 3\,7\,7\,2\,5\,4.$$

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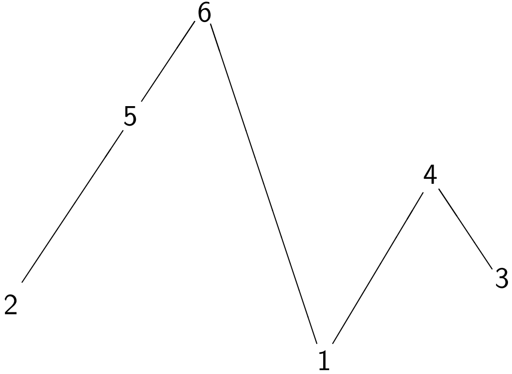

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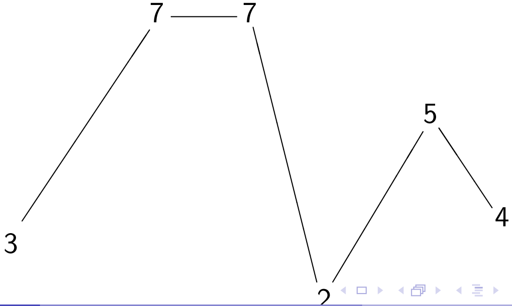

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Key Properties: If $\nabla_n(\sigma, \lambda) = w$, then

(1) $\mathrm{inv}\,\sigma + \mathrm{sum}\,\lambda = \mathrm{sum}\,w$ and, for $i < m$,

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Remark: ∇_n preserves general shape. As illustrations, (1) peaks are preserved as $\sigma_k < \sigma_{k+1} > \sigma_{k+2}$ if and only if $w_k \le w_{k+1} > w_{k+2}$ and (2) up-down permutations coincide with up-down compositions.

Definition: For $P \subseteq \bigcup_{m \ge 1} S_m$ and $w \in K_n$, set $P(w) = P(\sigma)$ where σ is the unique permutation satisfying $w = \nabla_n(\sigma, \lambda)$.

Theorem $\mathcal{PS} \subset \mathcal{PC}$

If $P \subseteq \bigcup_{m\geq 1} S_m$ and if $B_n \subseteq S_n$, then

$$\sum_{n\geq 0} \sum_{\sigma\in B_n} q^{\mathrm{inv}\,\sigma} \left(\prod_{p\in P} y_p^{p(\sigma)} \right) \frac{z^n}{(q;q)_n}$$
$$= \sum_{n\geq 0} \sum_{w\in \phi_n(B_n,\Lambda_n)} q^{\mathrm{sum\,w}} \left(\prod_{p\in P} y_p^{p(w)} \right) (z/q)^n.$$

Moreover, the above equality remains true if $y_p^{p(\sigma)}$ and $y_p^{p(w)}$ are respectively replaced by $y_p^{p_{no}(\sigma)}$ and $y_p^{p_{no}(w)}$ for some or all $p \in P$.

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$$\text{Gessel}: \quad \sum_{n\geq 0} \sum_{\sigma\in \mathrm{UD}S_n} \frac{q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \sec_q z + \tan_q z$$

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Mendes, Remmel : $\displaystyle\sum_{n\geq 0} \sum_{\sigma\in S_n} \frac{y^{\mathrm{pic}\,(\sigma)} q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan_q(z\sqrt{y-1})}$

Heubach, Mansour : $\displaystyle\sum_{n\geq 0} \sum_{w\in K_n} y^{\mathrm{pic}\,(w)} q^{\mathrm{sum}\,w} (z/q)^n = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan_q(z\sqrt{y-1})}$

3^{rd} Ex of Theorem $\mathcal{PS} \subset \mathcal{PC}$: For $P \subseteq S_m$ with $m > 1$,

$$\sum_{n\geq 0} \sum_{\sigma\in S_n} \frac{q^{\mathrm{inv}\,\sigma} y^{P_{no}(\sigma)} z^n}{(q;q)_n} = \frac{\mathcal{K}_q(z)}{1 - y + y\left(1 - z(1-q)^{-1}\right)\mathcal{K}_q(z)}$$

where $\mathcal{K}_q(z) = \sum_{n\geq 0} \left(\sum_{\sigma\in S_n} q^{\mathrm{inv}\,\sigma} 0^{P(\sigma)}\right) z^n/(q;q)_n$ is the q-exponential generating function for permutations that avoid P.

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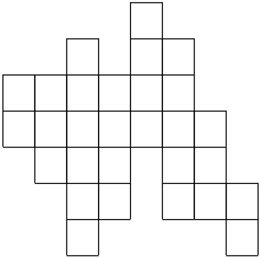
$$\sum_{n\geq 0}\sum_{w\in K_n} y^{P_{no}(w)} q^{\operatorname{sum} w} z^n = \frac{L_q(z)}{1 - y + y\left(1 - zq(1-q)^{-1}\right)L_q(z)}$$

where $L_q(z) = \sum_{n\geq 0}\left(\sum_{w\in K_n} q^{\operatorname{sum} w} 0^{P(w)}\right) z^n$ is the generating function for compositions that avoid P.

Remark: The latter is more and less general than a result due to Heubach, Kitaev, and Mansour; for a pattern set of cardinality 1, their result holds for an arbitrary alphabet of positive integers.

Consecutive Patterns in Column-Convex Polyominoes



Remark 1: The enumeration of CCPs and of subclasses of CCPs by various statistics (area, perimeter, column number, ...etc) has been widely studied. However, very little attention has been paid to ridge patterns in CCPs.

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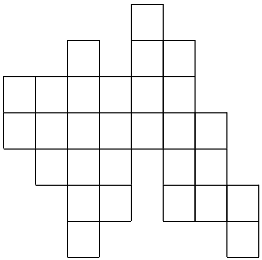

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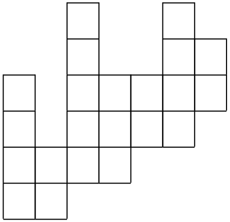
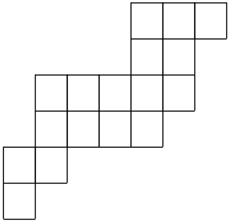
Remark 2: The simplest ridge patterns are formed between two adjacent columns. The two-column ridge patterns may be used to characterize many of the common classes of CCPs. For instance, a CCP with no lower descents is known as a directed column-convex polyomino (DCCP).

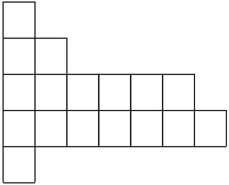
Common Classes of CCPs.



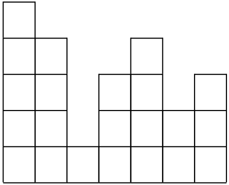
Directed Column–Convex Polyomino (DCCP): No lower descents



Parallelogram Polyomino: No lower or upper descents



Stack Polyomino: No upper ascents and no lower descents



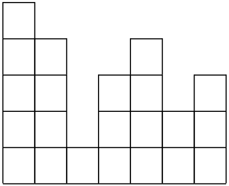
Wall Polyomino: No lower ascents and no lower descents

Compositions and Wall Polyominoes

Notation: Let WP_n= the set of wall polyominoes with n columns.

Bijection: Let γ_n denote the “natural” bijection from K_n to WP_n.

Example: γ_7 maps the composition $w = 5\,4\,1\,3\,4\,2\,3$ to



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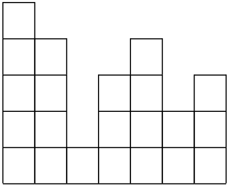

Properties: If $\gamma_n(w) = Q$, then

$$\mathrm{area}\, Q = \mathrm{sum}\, w \;\text{ and }\; \mathrm{per}\, Q - 2\mathrm{col}\, Q = \mathrm{var}\, w$$

where variation of w is $\mathrm{var}\, w = \sum_{k=0}^{n} |w_{k+1} - w_k|$ with the convention that $w_0 = 0 = w_{n+1}$.

Fact $\mathcal{PC} \subset \mathcal{PCCP}$

If P is a pattern set defined for compositions and if $B_n \subseteq K_n$, then

$$\sum_{n\geq 0}\sum_{w\in B_n} c^{\operatorname{var} w} q^{\operatorname{sum} w} z^n \prod_{p\in P} y_p^{p(w)} = \sum_{n\geq 0}\sum_{Q\in\gamma_n(B_n)} c^{\operatorname{per} Q} q^{\operatorname{area} Q} (z/c^2)^n \prod_{p\in P} y_p^{p(Q)}.$$

Example of Fact $\mathcal{PC} \subset \mathcal{PCCP}$

$$\mathrm{Raw, Tief}: \sum_{Q \in \mathrm{DCCP}} a_u^{\mathrm{uasc}\,Q} a_l^{\mathrm{lasc}\,Q} b_u^{\mathrm{ulev}\,Q} b_l^{\mathrm{llev}\,Q} c^{\mathrm{per}\,Q} d^{\mathrm{udes}\,Q} h^{\mathrm{relh}\,Q} q^{\mathrm{area}\,Q} z^{\mathrm{col}\,Q}$$

$$= \frac{c^2 h \sum_{n\geq 0} \frac{(c^2qz)^{n+1}}{1-c^2hq^{n+1}} \prod_{k=1}^{n} \left(b_l + \frac{a_l c^2 hq^k}{1-c^2hq^k}\right)\left(b_u + \frac{c^2dq^k}{1-c^2q^k} - \frac{a_u}{1-q^k}\right)}{1 - a_u \sum_{n\geq 1} \frac{(c^2qz)^n}{1-q^n} \prod_{k=1}^{n} \left(b_l + \frac{a_l c^2 hq^k}{1-c^2hq^k}\right) \prod_{k=1}^{n-1}\left(b_u + \frac{c^2dq^k}{1-c^2q^k} - \frac{a_u}{1-q^k}\right)}.$$

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Setting $a_l = 0$, $a_u = a$, $b_u = b$, $b_l = h = 1$, and replacing z by z/c^2 gives the gen func for compositions by ascents, levels, descents, and variation. ($c = 1$ is a classic result due to Carlitz)

$$\sum_{n \geq 0} \sum_{w \in K_n} a^{\text{asc}\, w} b^{\text{lev}\, w} d^{\text{des}\, w} c^{\text{var}\, w} q^{\text{sum}\, w} z^n$$

$$= 1 + \frac{c^2 \sum_{n \geq 0} \frac{(qz)^{n+1}}{1-c^2 q^{n+1}} \prod_{k=1}^{n} \left(b + \frac{c^2 dq^k}{1-c^2 q^k} - \frac{a}{1-q^k}\right)}{1 - a \sum_{n \geq 1} \frac{(qz)^n}{1-q^n} \prod_{k=1}^{n-1} \left(b + \frac{c^2 dq^k}{1-c^2 q^k} - \frac{a}{1-q^k}\right)}.$$

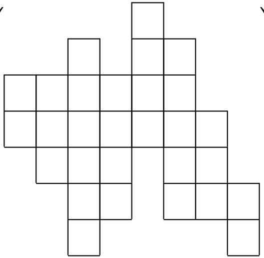
The Inclusion $\mathcal{PCCP} \subset \mathcal{PW}$

Let $X = \left\{ \binom{j}{m} : j, m \text{ are positive integers} \right\}$ and let

$$\mathcal{Y} = \bigcup_{n \geq 0} \left\{ \begin{pmatrix} j_1 \; j_2 \; \cdots \; j_n \\ m_1 \; m_2 \ldots m_n \end{pmatrix} \in X^n : m_n = 1 \text{ and } j_k + j_{k+1} > m_k \text{ for } 1 \leq k < n \right\}.$$

For a CCP Q with n columns, define $\delta(Q) = \binom{j_1 \; j_2 \; \cdots \; j_n}{m_1 \; m_2 \ldots m_n}$ where j_k is the number cells in Q_k, $m_n = 1$, and, for $1 \leq k < n$, m_k is the change in the y-ordinate from the bottom edge of $(k+1)^{st}$ column of Q to the top edge of the k^{th} column of Q.

$\delta\Big($



$\Big) = \binom{2\,3\,6\,4\,4\,5\,3\,2}{3\,5\,5\,2\,6\,5\,4\,1}$

Factors and Consecutive Patterns in Words

Let X^* denote the free moniod generated by the alphabet X.

Definition: For $\mathcal{F} \subseteq X^+$, a factor f of a word w is a said to be a consecutive $\mathcal{F}$-pattern in w if $f \in \mathcal{F}$. The number of consecutive $\mathcal{F}$-patterns in w is denoted by $\mathcal{F}(w)$.

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Definition: For $\mathcal{F} \subset X^+$, an $\mathcal{F}$-cluster is a triple (w, ν, β) in which

$$w = w_1 w_2 \dots w_{\operatorname{len} w} \in X^+,$$
$$\nu = (f_{(1)}, f_{(2)}, \dots, f_{(k)}) \text{ for some } k \geq 1 \text{ with each } f_{(i)} \in \mathcal{F}, \text{ and}$$
$$\beta = (b_1, b_2, \dots, b_k) \text{ with each } b_i \text{ being a positive integer}$$

where $f_{(i)} = w_{b_i} w_{b_i+1} \dots w_{b_i + \operatorname{len} f_{(i)} - 1}$, each $w_i w_{i+1}$ is a factor of some $f_{(j)}$, $b_1 \leq b_2 \leq \cdots \leq b_k$, and if $b_i = b_{i+1}$, then $\operatorname{len} f_{(i)} < \operatorname{len} f_{(i+1)}$.

Factors and Consecutive Patterns in Words

Let X^* denote the free moniod generated by the alphabet X.

Definition: For $\mathcal{F} \subseteq X^+$, a factor f of a word w is a said to be a consecutive $\mathcal{F}$-pattern in w if $f \in \mathcal{F}$. The number of consecutive $\mathcal{F}$-patterns in w is denoted by $\mathcal{F}(w)$.

Definition: For $\mathcal{F} \subset X^+$, an $\mathcal{F}$-cluster is a triple (w, ν, β) in which

$$\begin{aligned} &w = w_1 w_2 \dots w_{\operatorname{len} w} \in X^+, \\ &\nu = (f_{(1)}, f_{(2)}, \dots, f_{(k)}) \text{ for some } k \geq 1 \text{ with each } f_{(i)} \in \mathcal{F}, \text{ and} \\ &\beta = (b_1, b_2, \dots, b_k) \text{ with each } b_i \text{ being a positive integer} \end{aligned}$$

where $f_{(i)} = w_{b_i} w_{b_i+1} \dots w_{b_i + \operatorname{len} f_{(i)} - 1}$, each $w_i w_{i+1}$ is a factor of some $f_{(j)}$, $b_1 \leq b_2 \leq \cdots \leq b_k$, and if $b_i = b_{i+1}$, then $\operatorname{len} f_{(i)} < \operatorname{len} f_{(i+1)}$.

Definition: The cluster generating function over a subset $W \subset X^*$ is the formal series

$$C_{\mathcal{F}}(\mathbf{y}, W) = \sum_{(w,\nu,\beta) \in C_{\mathcal{F}},\ w \in W} \left(\prod_{f \in \mathcal{F}} y_f^{f(\nu)} \right) w$$

Words by Factors Theorem (Goulden and Jackson)

If, for nonempty $L, R \subseteq X$ and a nonempty $\mathcal{F} \subseteq X^+$, we define

$$\mathcal{L}(\mathbf{y}) = \sum_{l \in L} l + C_{\mathcal{F}}(\mathbf{y}, LX^*), \quad \mathcal{R}(\mathbf{y}) = \sum_{r \in R} r + C_{\mathcal{F}}(\mathbf{y}, X^*R), \quad \text{and}$$

$$\mathcal{X}(\mathbf{y}) = \sum_{x \in X} x + C_{\mathcal{F}}(\mathbf{y}, X^*)$$

and if the result of replacing y_f in $\mathbf{y}$ by $y_f - 1$ is denoted by $\mathbf{y} - \mathbf{1}$, then

$$\sum_{w \in X^*} \left(\prod_{f \in \mathcal{F}} y_f^{f(w)} \right) w = (1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1},$$

$$\sum_{w \in LX^*} \left(\prod_{f \in \mathcal{F}} y_f^{f(w)} \right) w = \mathcal{L}(\mathbf{y} - \mathbf{1})(1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1},$$

$$\sum_{w \in X^*R} \left(\prod_{f \in \mathcal{F}} y_f^{f(w)} \right) w = (1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1} \mathcal{R}(\mathbf{y} - \mathbf{1}), \quad \text{and}$$

$$\sum \left(\prod y_f^{f(w)} \right) w = C_{\mathcal{F}}(\mathbf{y} - \mathbf{1}, LX^*R) + \mathcal{L}(\mathbf{y} - \mathbf{1})(1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1} \mathcal{R}(\mathbf{y}$$

Application of Words by Factors Theorem to Permutations

Consider the alphabet $N = \{1, 2, 3, \ldots\}$, let $P \subseteq \bigcup_{m \geq 1} S_m$, and put

$$D_P(\mathbf{y}; z) = \sum_{(w,\nu,\beta) \in C_{\mathcal{F}_P}} \left(\prod_{p \in P} y_p^{p(\nu)} \right) q^{\mathrm{sum}\, w} z^{\mathrm{len}\, w} \quad \text{where} \quad p(\nu) = \sum_{f \in \mathcal{F}_p} f(\nu).$$

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Replacement of w by $q^{\operatorname{sum} w} z^{\operatorname{len} w}$ in the first identity of the Words by Factors Theorem and application of Fedou's bijection implies

Extension of Rawlings' Theorem: If $P \subseteq \bigcup_{m \geq 1} S_m$, then

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} \left(\prod_{p \in P} y_p^{p(\sigma)} \right) \frac{q^{\operatorname{inv} \sigma} z^n}{(q; q)_n} = \left(1 - z(1-q)^{-1} - D_P(\mathbf{y} - \mathbf{1}; z/q) \right)^{-1}.$$

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Non-overlapping Version: For $P \subseteq S_m$ with $m > 1$, then

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} y^{P_{no}(\sigma)} \frac{q^{\text{inv}\, \sigma} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - (1-y) D_P(-\mathbf{1}) \right)^{-1}.$$

Extension of Rawlings' Theorem

If $P \subseteq \bigcup_{m \geq 1} S_m$, then

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\left(\prod_{p\in P} y_p^{p(\sigma)}\right)\frac{q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - D_P(\mathbf{y}-\mathbf{1}; z/q)\right)^{-1}.$$

Ex 1: Permutations by Peaks

Let $\mathrm{pic} = \{132, 231\}$. As the pic-clusters are in 1-to-1 correspondence with the up-down compositions of odd length > 1,

$$\frac{z}{1-q} + D_{\mathrm{pic}}(y; z/q) = \frac{1}{\sqrt{y}}\sum_{n\geq 0}\sum_{w\in \mathrm{UD}K_{2n+1}} q^{\mathrm{sum}\,w}\left(\frac{z\sqrt{y}}{q}\right)^{2n+1} = \frac{\tan_q(z\sqrt{y})}{\sqrt{y}}.$$

Thus,

$$\text{Mendes, Remmel}: \sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^{\mathrm{pic}(\sigma)} q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan_q(z\sqrt{y-1})}$$

Ex 2: Permutations by Peaks and Twin Peaks

Let $\mathrm{tpic} = \{p \in S_5 : p_1 < p_2 > p_3 < p_4 > p_5\}$.

Result using extension of Rawlings' Theorem:

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{q^{\mathrm{inv}\,\sigma}x^{\mathrm{pic}\,(\sigma)}y^{\mathrm{tpic}\,(\sigma)}z^n}{(q;q)_n}=\frac{1}{1-\frac{z}{1-q}-\sum_{n\geq 1}A_n(x-1,y-1)B_n(q)(z/q)^{2n+1}}$$

where $A_n(x,y)=(xz+yz^2+xyz^2)(1-xz-xyz-xyz^2-yz-yz^2)^{-1}\big|_{z^n}$ and $B_n(q)=\tan_q z\big|_{z^{2n+1}}$.

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where $A_n(x,y) = (xz+yz^2+xyz^2)(1-xz-xyz-xyz^2-yz-yz^2)^{-1}\big|_{z^n}$ and $B_n(q) = \tan_q z\big|_{z^{2n+1}}$.

Alternative Result using Pattern Algebra of Goulden and Jackson:

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{q^{\mathrm{inv}\,\sigma}x^{\mathrm{pic}\,(\sigma)}y^{\mathrm{tpic}\,(\sigma)}z^n}{(q;q)_n}$$

$$= \left(1-\frac{s_+\sin_q(z\sqrt{r_+})}{2\sqrt{r_+}\cos_q(z\sqrt{r_+})}-\frac{s_-\sin_q(z\sqrt{r_-})}{2\sqrt{r_-}\cos_q(z\sqrt{r_-})}\right)^{-1}$$

where $r_\pm = (xy-1\pm\sqrt{D})/2$, $s_\pm = 1\pm(2x-xy-1)/\sqrt{D}$, and $D = (xy+1)^2-4x$.

q-Olivier functions

$$\Phi_{j,k}(z) = \sum_{n \geq 0} \frac{z^{jn+k}}{(q;q)_{jn+k}}$$

Examples: $\Phi_{1,0}(z) = e_q(z), \quad \Phi_{2,0}(iz) = \cos_q z, \quad$ and $\Phi_{2,1}(iz) = i \sin_q z$.

Ex 3: Permutations by (i,d)-peaks and Inversions

Let $P_{i,d} = \{p \in S_{i+d-1} : p_1 < p_2 < \cdots < p_i > p_{i+1} > \cdots > p_{i+d-1}\}$.

Example of a (3,3)-peak:

$p = 2\,4\,5\,3\,1 =$

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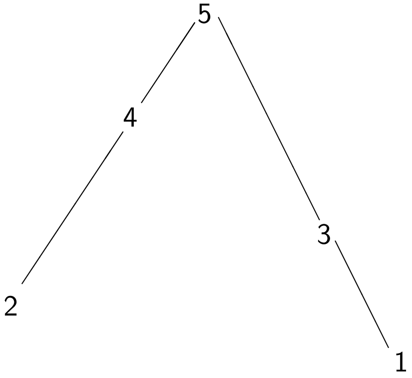

Ex 3: Permutations by (i,d)-peaks and Inversions

If, for $i, j, d \geq 2$, we set $\mu = i + d - 2$ and $\xi_m = \sqrt[m]{-1}$, then

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{y^{P_{i,d}(\sigma)} q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - \frac{K_{i,i,d;1}(\sqrt[\mu]{y-1}\,z)}{\sqrt[\mu]{y-1}}\right)^{-1}$$

where, for $k \geq 1$,

$$K_{i,j,d;k}(z) = \sum_{m\geq 0} \sum_{w\in K_{i,d;(j,d)^m;k}} q^{\mathrm{sum}\,w} z^{\mathrm{len}\,w}.$$

Ex 3: Permutations by (i,d)-peaks and Inversions

If, for $i, j, d \geq 2$, we set $\mu = i + d - 2$ and $\xi_m = \sqrt[m]{-1}$, then

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where, for $k \geq 1$,

$$K_{i,j,d;k}(z) = \sum_{m\geq 0} \sum_{w\in K_{i,d;(j,d)^m;k}} q^{\mathrm{sum}\,w} z^{\mathrm{len}\,w}.$$

Moreover, $K_{i,j,d;k}(z)$ satisfies, for $d \geq 3$ and $\nu = j + d - 2$, the recurrence

$$K_{i,j,d;k}(z) = \frac{\xi_\nu^{-\mu} K_{i,j+1,d-1;1}(\xi_\nu z)\big(z^k (q;q)_k^{-1} + \xi_\nu^{-k} K_{j,j+1,d-1;k+1}(\xi_\nu z)\big)}{1 + K_{j,j+1,d-1;1}(\xi_\nu z)}$$

$$- \xi_\nu^{-\mu-k} K_{i,j+1,d-1;k+1}(\xi_\nu z)$$

with the initial condition

$$K_{i,j,2;k}(z) = \xi_j^{-i-k}\,[-\Phi_{j,i+k}(\xi_j z) + \Phi_{j,i}(\xi_j z)\Phi_{j,k}(\xi_j z)/\Phi_{j,0}(\xi_j z)].$$

Permutations by (3,3)-peaks and Inversions

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^{P_{3,3}(\sigma)}q^{\mathrm{inv}\,\sigma}z^n}{(q;q)_n}=\left(1-z(1-q)^{-1}-\frac{K_{3,3,3;1}(\sqrt[4]{y-1}\,z)}{\sqrt[4]{y-1}}\right)^{-1}$$

where

$$K_{3,3,3;1}(z)=\frac{-K_{3,4,2;1}(\xi_4 z)\left(z(1-q)^{-1}+\xi_4^{-1}K_{3,4,2;2}(\xi_4 z)\right)}{1+K_{3,4,2;1}(\xi_4 z)}+\xi_4^{-1}K_{3,4,2;2}(\xi_4 z)$$

with

$$K_{3,4,2;1}(z)=-\frac{\Phi_{4,3}(\xi_4 z)\Phi_{4,1}(\xi_4 z)}{\Phi_{4,0}(\xi_4 z)}+\Phi_{4,4}(\xi_4 z)\quad\text{and}$$

$$K_{3,4,2;2}(z)=\xi_4^{-1}\left[-\frac{\Phi_{4,3}(\xi_4 z)\Phi_{4,2}(\xi_4 z)}{\Phi_{4,0}(\xi_4 z)}+\Phi_{4,5}(\xi_4 z)\right].$$

Ex 4: Permutations by m-Peak Ranges of (i, d)-Peaks

Corollary

If $i, d \geq 2$, $m \geq 1$, and $\nu = i + d - 2$, then the generating function for permutations by uniform m-peak ranges and inversions is

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} \frac{y^{P_{(i,d)^m}(\sigma)} q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \left(1 - \frac{z}{1-q} - \sum_{n \geq m} A_{n,m}(y-1) B_n(q) z^{n\nu+1}\right)^{-1}$$

where

$$A_{n,m}(y) = \left.\frac{yz^m(1-z)}{1-z-yz(1-z^m)}\right|_{z^n} \quad \text{and} \quad B_n(q) = K_{i,i,d;1}(z)|_{z^{n\nu+1}}$$

with $K_{i,i,d;1}(z)$ as determined earlier.

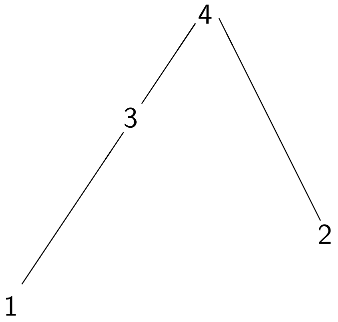
Remark: For $i = d = 2$ with $y = 0$, the above provides a solution to a problem posed by Kitaev of counting permutations that avoid $(2m+1)$-reverse-alternating patterns.

Ex 5: Permutations by (i,m)-Maxima and Inversions

Let $p_{(m)} \in S_{i+1}$ with $p_{(m)1} < p_{(m)2} < \cdots < p_{(m)i}$ and $p_{(m)i+1} = i + 1 - m$.

Example: The (3,2)-maxima pattern in S_{3+1}:

$p_{(2)} = 1\,3\,4\,2 =$



Remark: Carlitz and Scoville referred to $(2, 1)$ and $(2, 2)$-maxima as rising and falling maxima.

Ex 5: Permutations by (i,m)-Maxima and Inversions

Corollary (Words by Factors via extension of Rawlings' Theorem): If $i \geq 2$, $1 \leq m \leq i$ and $\xi_i = \sqrt[i]{-1}$, then

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\left(\prod_{m=1}^{i} y_m^{p_{(m)}(\sigma)}\right)\frac{q^{\mathrm{inv}\,\sigma}z^n}{(q;q)_n} = \left(1-\frac{\Phi_{i,1}(\mathbf{y}-\mathbf{1};\xi_i z)}{\xi_i\Phi_{i,0}(\mathbf{y}-\mathbf{1};\xi_i z)}\right)^{-1} \text{ where}$$

$$\Phi_{i,k}(y_1,\ldots,y_i;z)=\sum_{n\geq 0}\frac{z^{in+k}}{(q;q)_{in+k}}\prod_{j=0}^{n-1}\left(y_i+\sum_{m=1}^{i-1}(y_i-y_m)q^m\begin{bmatrix} ij+k+m-1 \\ m\end{bmatrix}\right).$$

For $i=2$, $y_1=y$, and $y_2=1$, above gives Mendes and Remmel's q-analog of Elizalde and Noy's result for permutations by $p=132$:

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^{132(\sigma)}q^{\mathrm{inv}\,\sigma}z^n}{(q;q)_n} = \left(1-\sum_{n\geq 0}\frac{(y-1)^n q^n z^{2n+1}}{(q^2;q^2)_n(1-q^{2n+1})(1-q)^n}\right)^{-1}.$$

An Aside

Proof of the generating function for permutations by (i, m)-maxima of the previous slide reveals the generating function for up-down permutations of type $(i, i, 2; 1)$ by (i, m)-maxima:

$$\frac{z}{1-q} + \sum_{\sigma \in \mathrm{UD}S_{i,i,2;1}} \left(\prod_{m=1}^{i} y_m^{p_{(m)}} \right) \frac{q^{\mathrm{inv}\,\sigma} z^{\mathrm{len}\,\sigma}}{(q;q)_{\mathrm{inv}\,\sigma}} = \frac{\Phi_{i,1}(y_1, \ldots, y_{i-1}, 1; \xi_i z)}{\xi_i \Phi_{i,0}(y_1, \ldots, y_{i-1}, 1; \xi_i z)}.$$

Setting $y_1 = y_2 = ... = y_i = 1$, replacing z by $(1-q)z$, and letting $q \to 1$ gives a result of Carlitz's .

Ex 5: Perms by (i,m)-Maxima using the Temperley Method

If $i \geq 2$ and $1 \leq m \leq i$, then the generating function for permutations by (i,m)-maxima and inversions is also given by

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\left(\prod_{m=1}^{i} y_m^{p_{(m)}(\sigma)}\right)\frac{q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n}$$

$$=\left(1-\frac{\displaystyle\sum_{n\geq 0}\frac{z^{in+1}}{1-q^{in+1}}\prod_{k=0}^{n-1}T(q^{ik})}{\displaystyle 1-\frac{y_i-1}{(q;q)_{i-1}}\sum_{n\geq 1}\frac{z^{in}}{1-q^{in}}\prod_{k=1}^{n-1}T(q^{ik-1})}\right)^{-1}$$

where

$$T(b)=\sum_{m=1}^{i-1}\frac{(y_m-1)q^m}{(q;q)_m(q^{m+1}b;q)_{i-m}}-\frac{y_i-1}{(q;q)_{i-1}(1-qb)}.$$

Ex 6: Permutations by maximal number of non-overlapping $P = \{1243, 1342, 1432, 2341, 2431, 3421\}$

Corollary

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{y^{P_{no}(\sigma)} q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \frac{\mathcal{K}}{1-y+y(1-\frac{z}{1-q})\mathcal{K}}$$

where

$$\mathcal{K} = \frac{e_q(z)e_q(-z)+\cos_q^2 z+\sin_q^2 z+2e_q(-z)\cos_q z + (e_q(z)+e_q(-z))\sin_q z}{4e_q(-z)\cos_q z}.$$

Remark: $\mathcal{K}$ is a q-analog of Kitaev's generating function that enumerates permutations that avoid $P = \{4312, 4213, 4123, 3214, 3124, 2134\}$.

Ex 7: DCCPs by All Five Two-Column Statistics

$$\text{Raw, Tief} : \sum_{Q \in \text{DCCP}} a_u^{\text{uasc}\, Q} a_l^{\text{lasc}\, Q} b_u^{\text{ulev}\, Q} b_l^{\text{llev}\, Q} c^{\text{per}\, Q} d^{\text{udes}\, Q} h^{\text{relh}\, Q} q^{\text{area}\, Q} z^{\text{col}\, Q}$$

$$= \frac{c^2 h \sum_{n \geq 0} \frac{(c^2 qz)^{n+1}}{1 - c^2 h q^{n+1}} \prod_{k=1}^{n} \left(b_l + \frac{a_l c^2 h q^k}{1 - c^2 h q^k} \right) \left(b_u + \frac{c^2 d q^k}{1 - c^2 q^k} - \frac{a_u}{1 - q^k} \right)}{1 - a_u \sum_{n \geq 1} \frac{(c^2 qz)^n}{1 - q^n} \prod_{k=1}^{n} \left(b_l + \frac{a_l c^2 h q^k}{1 - c^2 h q^k} \right) \prod_{k=1}^{n-1} \left(b_u + \frac{c^2 d q^k}{1 - c^2 q^k} - \frac{a_u}{1 - q^k} \right)}.$$

Ex 8: CCPs by All Six Two-Column Statistics (Temperley)

If we set $F(x)$ equal to

$$\sum_{Q \in \mathrm{CCP}} a_u^{\mathrm{uasc}\,Q} b_u^{\mathrm{ulev}\,Q} d_u^{\mathrm{udes}\,Q} a_l^{\mathrm{lasc}\,Q} b_l^{\mathrm{llev}\,Q} d_l^{\mathrm{ldes}\,Q} c^{\mathrm{per}\,Q} q^{\mathrm{area}\,Q} h^{\mathrm{relh}\,Q} x^{\alpha(Q)} z^{\mathrm{col}\,Q}$$

where $\alpha(Q)$ denotes the area of the last column in Q, then

$$F(x) = \frac{\begin{vmatrix} R(x) & S(x) & T(x) \\ R(1) & S(1)-1 & T(1) \\ R(\frac{1}{h}) & S(\frac{1}{h}) & T(\frac{1}{h})-1 \end{vmatrix}}{\begin{vmatrix} S(1)-1 & T(1) \\ S(\frac{1}{h}) & T(\frac{1}{h})-1 \end{vmatrix}},$$

where

$$\begin{aligned} R(x) &= \sum_{n\geq 0} z^{n+1} y(x) y(qx) \dots y(q^{n-1}x) r(q^n x), \\ S(x) &= \sum_{n\geq 0} z^{n+1} y(x) y(qx) \dots y(q^{n-1}x) s(q^n x), \\ T(x) &= \sum z^{n+1} y(x) y(qx) \dots y(q^{n-1}x) t(q^n x), \text{ and} \dots \end{aligned}$$

Ex 8 Continued : The Rest of the Formula

$$
\begin{aligned}
r(x) &= \frac{qxc^4h}{1-qxc^2h}, \\
s(x) &= \frac{q^2x^2c^4ha_ua_l}{(1-qx)(1-qxc^2h)} + \frac{qxc^2a_lb_u}{1-qx} - \frac{qxc^2d_ua_l}{(1-h)(1-qx)}, \\
t(x) &= \frac{qxc^2hd_ub_l}{1-qxh} + \frac{q^2x^2c^4hd_ud_l}{(1-qxh)(1-qxc^2)} + \frac{qxc^2h^2d_ua_l}{(1-h)(1-qxh)}, \\
y(x) &= \frac{qxc^4ha_ub_l}{1-qxc^2h} + \frac{q^2x^2c^6ha_ud_l}{(1-qxc^2)(1-qxc^2h)} - \frac{qxc^4ha_ua_l}{(1-qx)(1-qxc^2h)} \\
&\quad + c^2b_ub_l + \frac{qxc^4b_ud_l}{1-qxc^2} - \frac{c^2b_ua_l}{1-qx} - \frac{c^2d_ub_l}{1-qxh} \\
&\quad - \frac{qxc^4d_ud_l}{(1-qxh)(1-qxc^2)} + \frac{c^2d_ua_l}{(1-qx)(1-qxh)}.
\end{aligned}
$$

Ex 9: DCCPs by Upper Valleys

A column-segment $Q_k Q_{k+1} Q_{k+2}$ in a column-convex polyomino Q is said to be a valley provided that $Q_k Q_{k+1}$ is an upper descent and $Q_{k+1} Q_{k+2}$ is an upper ascent or an upper level.

Corollary of Words by Factors

$$\sum_{Q \in \mathrm{DCCP}} y^{\mathrm{val}(Q)} q^{\mathrm{area}\, Q} z^{\mathrm{col}\, Q}$$

$$= \frac{\displaystyle\sum_{n \geq 0} \frac{(1-y)^n q^{(n+1)(2n+1)} z^{2n+1}}{(q;q)_{2n+1}(q;q)_{2n}}}{\displaystyle\sum_{n \geq 0} \frac{(1-y)^n q^{n(2n+1)} z^{2n}}{(q;q)_{2n}^2} - \sum_{n \geq 0} \frac{(1-y)^n q^{(n+1)(2n+1)} z^{2n+1}}{(q;q)_{2n+1}^2}}.$$

Ex 10: CCPs by Peaks, Area, and Column Number (Temperley)

$$\sum_{Q \in \mathrm{CCP}} y^{\mathrm{pic}\,(Q)} q^{\mathrm{area}\,Q} z^{\mathrm{col}\,Q} = \frac{\left(\frac{zq}{1-q} + \frac{2z^2q^3}{(1-q)^3}\right)\left(1 + \frac{2zq}{(1-q)^2}\right)}{\left(1 - \frac{zq^2}{(1-q)^2}\right)\left(1 + \frac{zq}{(1-q)^2}\right) - \frac{2yz^2q^3}{(1-q)^4}}.$$