Pattern Avoiding Colored Partitions

Adam M. Goyt
Minnesota State University Moorhead
goytadam@mnstate.edu

Lara K. Pudwell
Valparaiso University
Lara.Pudwell@valpo.edu

Valparaiso University

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1. History and Definitions
2. Colored Partitions and Avoidance
3. A Flavor of the Proofs
4. Summary and Future Ideas

Outline

Who and What

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- The notion of signed set partitions was considered by Anders Björner and Michelle Wachs [1] from a poset and homological perspective.
- Now, we consider Pattern Avoidance in Colored Set Partitions. (Boom?)

Set Partition Definition

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Example

$137/25/46 \vdash [7]$

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Given any word $w \in [k]^n$ we may canonize w by replacing all occurrences of the first letter of w by 1, all occurrences of the next occurring letter by 2, etc.

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There is a bijection between all canonized words of length n and partitions of $[n]$.

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137/25/46 *corresponds to* 1213231.

We will say that a partition, π is of length n, $\ell(\pi) = n$, if its associated canonical word has n letters.

From now on we will refer to these canonical words as partitions.

Pattern Avoidance

Definition

Let σ be a partition with $\ell(\sigma) = n$ and π be a partition with $\ell(\pi) = k$. We say that σ contains π if there is a subsequence of σ of length k whose canonization is π. Otherwise we say that σ avoids π.

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Let $\sigma = 1213431$.

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Example

Let $\sigma = 1213431$.

σ *contains a copy of* 112 *namely* 1213431 *or* 1213431. *However,* σ *avoids* 1112.

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Definition

Denote the set of all k-colored set partitions of $[n]$ *by* $\Pi_n \wr C_k$.

Example

Consider $\sigma = 1213431 \in \Pi_7$ *from the previous slide. We can make* σ *an element of* $\Pi_7 \wr C_3$ *simply by choosing one of three colors for each of the elements. So* $1211341 \in \Pi_7 \wr C_3$.

Avoiding Colored Partitions

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We say that $\sigma \in \Pi_n \wr C_k$ contains a copy of $\pi \in \Pi_m \wr C_j$ if

1. *the uncolored version of σ contains a copy of the uncolored version of π and*

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Consider $\sigma = 1211341$. Then σ contains a copy of 122, but σ avoids 122.

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Consider $\sigma = 1211341$. Then σ contains a copy of 122, but σ avoids 122.

Definition

For a set of colored set partitions S let $\Pi_n \wr C_k(S)$ be the set of partitions in $\Pi_n \wr C_k$ that avoid every pattern in S.

Outline

Friendly Results

Theorem

For $n \geq 1$ and $c \geq 2$, $|\Pi_n \wr C_2(11, 11)| = \sum_{i=1}^{n} 2^i S(n, i)$ (OEIS A001861)

Theorem

For $n \geq 1$ and $c \geq 2$, $|\Pi_n \wr C_2(12)| = (B_{n+1} - B_{n+1}) - (B_{n+1} - B_n)$ (OEIS A011965)

$|\Pi_n \wr C_k(112)|$

Theorem

For $n \geq 1$ and $c \geq 2$, $|\Pi_n \wr C_k(112)| =$

$$B(n)(k-1)^n + \sum_{m=1}^{n}\sum_{j=1}^{m}\binom{n}{m}\binom{m}{j}B(n-j)(k-1)^{n-j} +$$

$$\sum_{1\leq i<j\leq n}\sum_{a,b}\sum_{d,e}\sum_{f,g}\sum_{m}\sum_{p,q}\sum_{\ell}\binom{i-1}{a,b}\binom{j-i-1}{d,e}\binom{n-j}{f,g}\cdot$$

$$\binom{i-a-b-1}{m}\binom{j-i-d-e-1}{p}\binom{n-a-b-f-g-j+i-m-1}{q}\cdot$$

$$S(p+q,\ell)m^{\ell}B(n-a-b-d-e-f-g-m-p-q-2)\cdot$$

$$((k-1)^b + b(k-1)^{b-1})((k-1)^a + a(k-1)^{a-1})(k-2)^{d+p}k^e\cdot$$

$$(k-1)^{n-a-b-d-e-m-p-2}.$$

Back

Avoiding 112 – Sketch of Proof

The Proof of this Theorem can be broken into 3 cases.

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The Proof of this Theorem can be broken into 3 cases.

Case 1: No elements are colored blue.

Case 2: Exactly one block contains elements colored blue.

Case 3: At least two blocks contain elements colored blue.

No elements are colored blue.

In this case there can't possibly be a copy of 112.

- $B(n)$ (n^{th} Bell number) ways to partition the elements in $[n]$
- $(k-1)^n$ ways to color each element any color but blue.

Thus there are $B(n)(k-1)^n$ such 112 avoiding partitions in $\Pi_n \wr C_k$.

Exactly one block contains elements colored blue.

In this case there can't possibly be a copy of 112.

- Select m elements to be in the block with the elements that are colored blue.
- Select j of these elements to be colored blue.
- Partition the remaining $n-m$ elements in $B(n-m)$ ways.
- Color the non-blue elements in $(k-1)^{n-j}$ ways.

Thus there are

$$\sum_{m=1}^{n}\sum_{j=1}^{m}\binom{n}{m}\binom{m}{j}B(n-m)(k-1)^{n-j}$$

such partitions avoiding 112.

The Final Case

At least two blocks contain elements colored blue and there is no copy of 112.

SEE BOARD!

Jump to theorem

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- Set Partition Statistics

Thank You

THANK YOU

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