Simple permutations poset

Adeline Pierrot   Dominique Rossin

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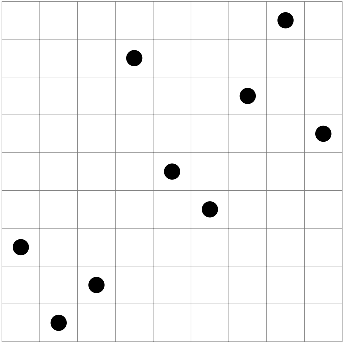
Patterns in permutations

Pattern relation $\preccurlyeq$:
$\pi \in S_k$ is a pattern of $\sigma \in S_n$ if
$\exists\ 1 \leq i_1 < \ldots < i_k \leq n$ such that
$\sigma_{i_1} \ldots \sigma_{i_k}$ is order-isomorphic to π.
We write $\pi \preccurlyeq \sigma$.

Example: $1\,3\,2\,4 \preccurlyeq 3\,1\,2\,8\,5\,4\,7\,9\,6$
since $2\,5\,4\,9 \equiv 1\,3\,2\,4$.



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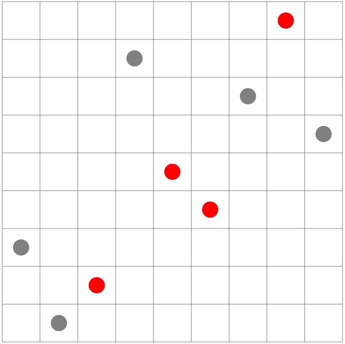
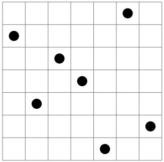
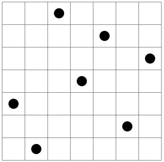


Interval = window of elements of σ whose values form a range
Example: 5 7 4 6 is an interval of 2 5 7 4 6 1 3

Simple permutation = has no interval except $1, 2, \ldots, n$ and σ
Equivalently: In the graphical representation, every non trivial bounding box has at least a point on his side.





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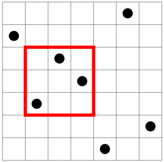

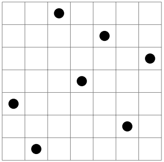

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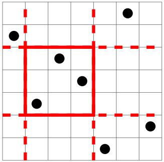

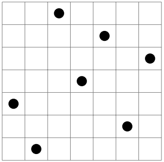

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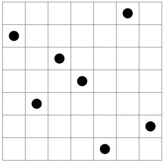

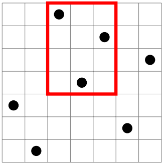

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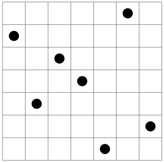

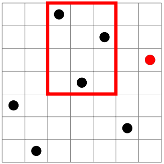

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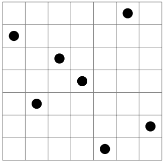

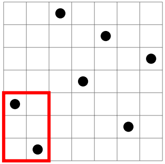

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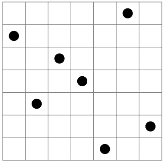

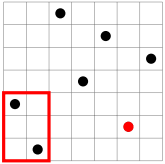

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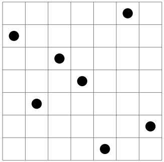

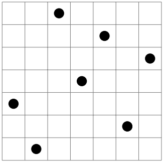

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Simple permutations and patterns

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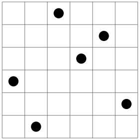

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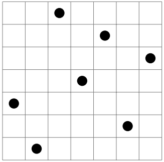

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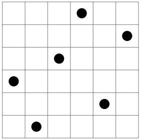

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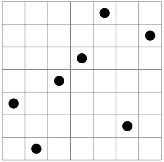

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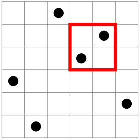

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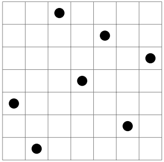

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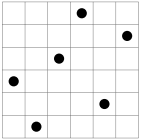

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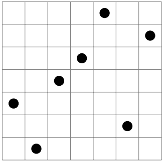

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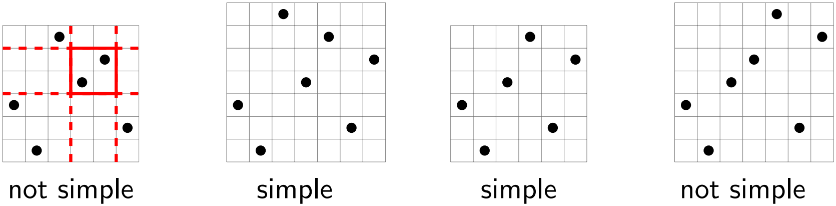

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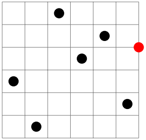

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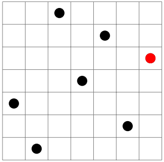

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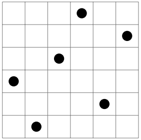

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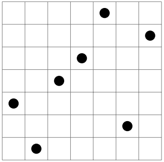

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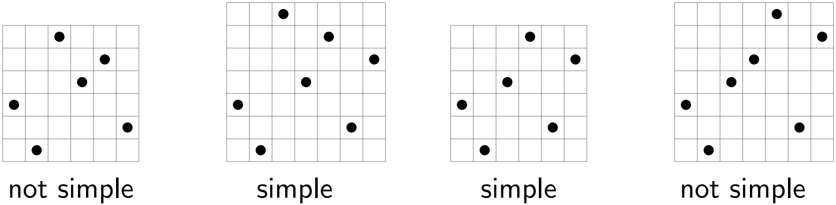

Example: $3\,1\,6\,4\,5\,2$ (not simple) $\preceq$ $3\,1\,7\,4\,6\,2\,5$ (simple)

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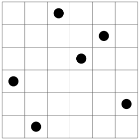

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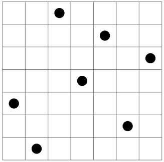

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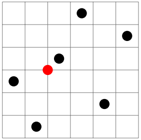

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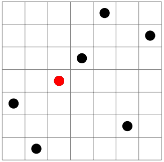

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Example: 3 1 6 4 5 2 (not simple) $\preceq$ 3 1 7 4 6 2 5 (simple)

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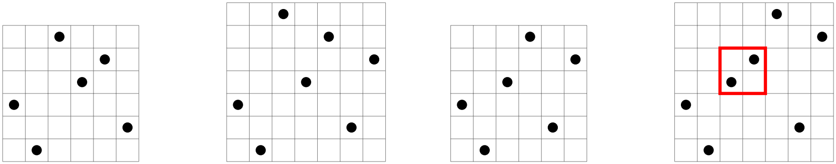

not simple | simple | simple | not simple

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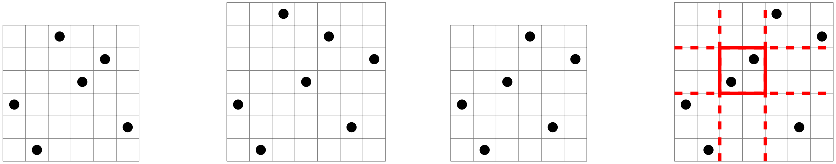

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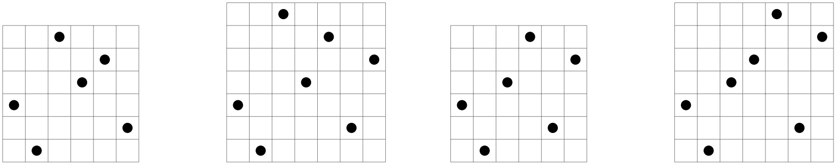

not simple — simple — simple — not simple

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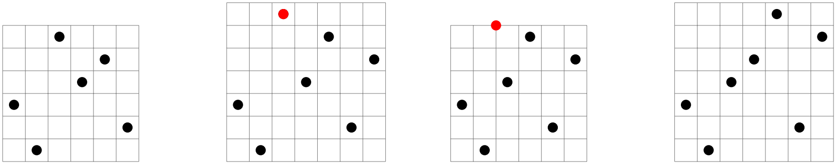

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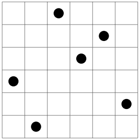

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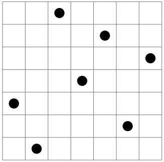

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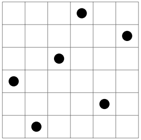

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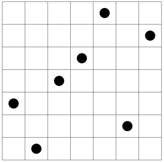

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Exceptional permutations

Definition: Exceptional permutations are simple permutations defined below for every $m \geq 2$:

- $2\ 4\ 6\ 8 \ldots (2m)\ 1\ 3\ 5 \ldots (2m-1)$
- $(2m-1)\ (2m-3) \ldots 1\ (2m)\ (2m-2) \ldots 2$
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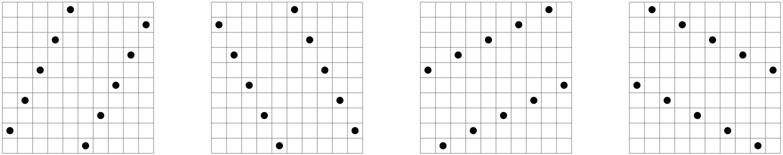

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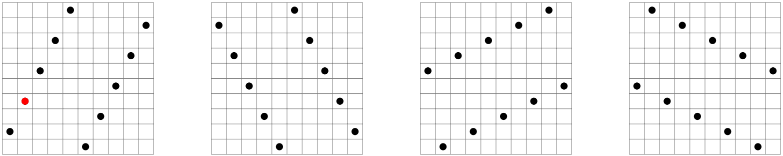

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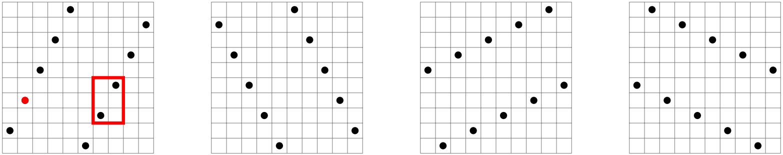

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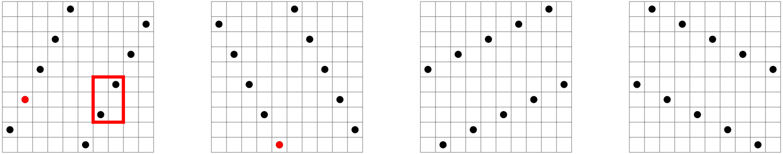

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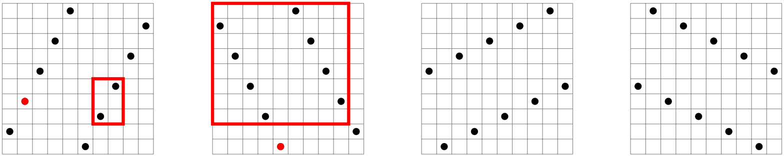

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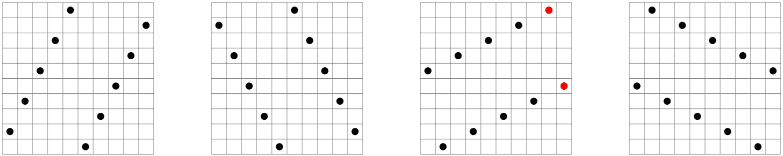

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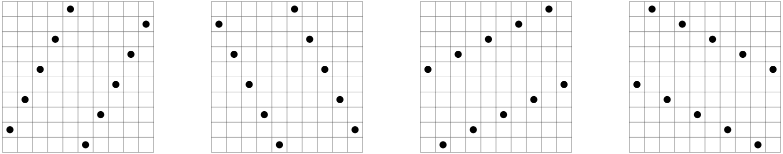
- $(2m-1)\ (2m-3) \ldots 1\ (2m)\ (2m-2) \ldots 2$
- $(m+1)\ 1\ (m+2)\ 2 \ldots (2m)\ m$
- $m\ (2m)\ (m-1)\ (2m-1) \ldots 1\ (m+1)$



Exceptional permutation of type 1, 2, 3 and 4

Why exceptional ?

Proposition: σ a non exceptional simple permutation, $4 \leq m \leq |\sigma|$
$\Rightarrow \exists$ a simple permutation π of size m such that $\pi \preceq \sigma$.

Proposition: σ an exceptional permutation $\Rightarrow \forall$ m such that $4 \leq m \leq |\sigma|$:

- m odd $\Rightarrow \sigma$ has no simple pattern of size m.
- m even $\Rightarrow \sigma$ has exactly one simple pattern of size m : the exceptional permutation of the same type as σ.

Constrained patterns (1)

$$\pi \preceq \sigma \xrightarrow{?} \pi \preceq \tau \preceq \sigma$$

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$$\pi \preceq \sigma \xrightarrow{?} \pi \preceq \tau \preceq \sigma$$

Proposition : π, σ two simple permutations, $3 \leq |\pi| \leq |\sigma| - 2 \Rightarrow \exists$ a simple permutation τ such that $\pi \leq \tau \leq \sigma$ and $|\tau| = |\pi| + 2$.

$$\underbrace{\pi \preceq \sigma}_{\geq 2} \Rightarrow \underbrace{\pi \preceq \tau}_{2} \preceq \sigma$$

Constrained patterns (2)

Proposition : σ a non exceptional simple permutation, $|\sigma| = n \geq 4$ and π a simple permutation, $|\pi| = n-2$, $\pi \preceq \sigma \Rightarrow \exists$ a simple permutation τ of size $n-1$ such that $\pi \preceq \tau \preceq \sigma$.

$$\sigma \text{ non exceptional, } \underbrace{\pi \preceq \sigma}_{2} \Rightarrow \underbrace{\pi \preceq \tau}_{1}\underbrace{\preceq \sigma}_{1}$$

Proof (Lemma)

Lemma : τ a non simple permutation such that $\tau \setminus \{\tau_i\}$ is simple. Then τ_i belongs to an interval of size 2 of τ or is in a corner of the graphical representation of τ.

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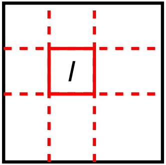

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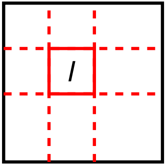

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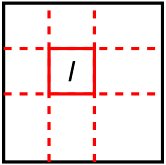

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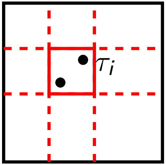

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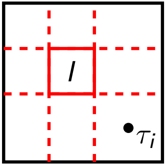

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$\tau \setminus \{\tau_i\}$ simple $\Rightarrow$ $I \setminus \{\tau_i\}$ is a trivial interval of τ

2 cases :

- $|I| = 2$ and τ_i belongs to I
- τ_i is the only point of τ outside I.

Proof (1)

Proposition : σ non exceptional, $\underbrace{\pi \preceq \sigma}_{2} \Rightarrow \underbrace{\pi \preceq \tau}_{1}\underbrace{\preceq \sigma}_{1}$

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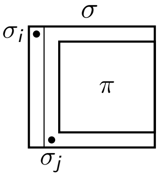
Suppose that such a permutation τ does not exists.
Let i, j such that $\pi = \sigma \setminus \{\sigma_i, \sigma_j\}$.
Then $\sigma \setminus \{\sigma_i\}$ is not simple but π is simple so σ_j belongs to an interval of size 2 of $\sigma \setminus \{\sigma_i\}$ or is in a corner of the graphical representation of $\sigma \setminus \{\sigma_i\}$



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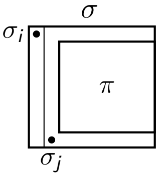

There are 3 different cases:

- σ_i and σ_j are both in a corner thanks to π. In that case π is a non trivial interval of σ, which contradicts the fact that σ is simple.
- σ_i belongs to an interval I of size 2 of $\sigma \setminus \{\sigma_j\}$ and σ_j is in a corner thanks to π (or exchange i and j).
- σ_i belongs to an interval I of size 2 of $\sigma \setminus \sigma_j$ and σ_j belongs to an interval J of size 2 of $\sigma \setminus \sigma_i$.

Proof (case 2)

- σ_i belongs to an interval of size 2 of $\sigma \setminus \{\sigma_j\}$ and σ_j is in a corner thanks to $\pi = \sigma \setminus \{\sigma_i, \sigma_j\}$.

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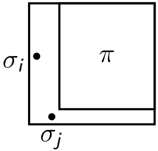

σ is simple $\Rightarrow$ σ_j is not in a corner of σ, but is in a corner of $\sigma \setminus \{\sigma_i\}$ $\Rightarrow$ σ_i is the only point separating σ_j from a corner.

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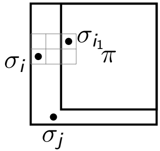
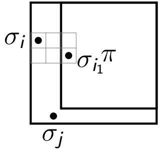


σ_i belongs to an interval $I = \{i, i_1\}$ of $\sigma \setminus \{\sigma_j\}$.



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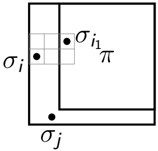
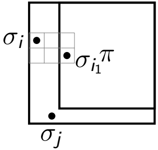


σ_i belongs to an interval $I = \{i, i_1\}$ of $\sigma \setminus \{\sigma_j\}$.

$\pi = \sigma \setminus \{\sigma_{i_1}, \sigma_j\}$ is simple but $\sigma \setminus \{\sigma_{i_1}\}$ is not simple



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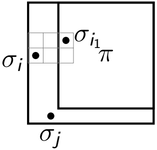

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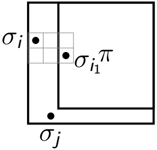
$\pi = \sigma \setminus \{\sigma_{i_1}, \sigma_j\}$ is simple but $\sigma \setminus \{\sigma_{i_1}\}$ is not simple $\Rightarrow$ σ_j belongs to an interval J of size 2 of $\sigma \setminus \{\sigma_{i_1}\}$ or is in a corner of $\sigma \setminus \{\sigma_{i_1}\}$, which is impossible.



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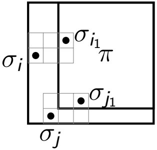

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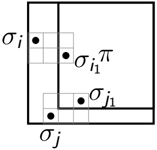

Let j_1 such that $J = \{j, j_1\}$, then $\pi = \sigma \setminus \{\sigma_{i_1}, \sigma_{j_1}\}$ is simple but $\sigma \setminus \{\sigma_{j_1}\}$ is not simple.

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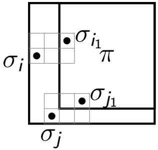

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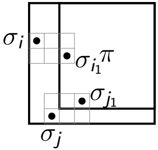

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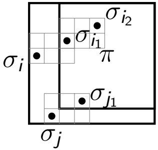

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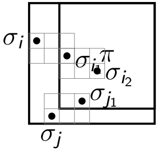

Let j_1 such that $J = \{j, j_1\}$, then $\pi = \sigma \setminus \{\sigma_{i_1}, \sigma_{j_1}\}$ is simple but $\sigma \setminus \{\sigma_{j_1}\}$ is not simple.
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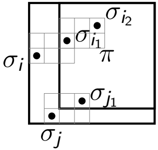

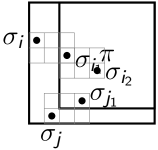

Let $i_0 = i$ and $j_0 = j$, we recursively build $i_0, j_0, i_1, j_1, \ldots$ such that $\forall k, \pi = \sigma \setminus \{\sigma_{i_k}, \sigma_{j_k}\} = \sigma \setminus \{\sigma_{j_k}, \sigma_{i_{k+1}}\}$ and $\sigma \setminus \sigma_{i_k}$ and $\sigma \setminus \sigma_{j_k}$ are not simple.

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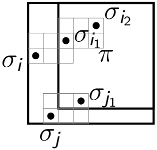

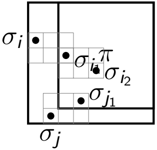

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Positions of σ_{i_k} and σ_{j_k} are fixed for all k as σ_{i_k} does not separate $\sigma_{i_{k-1}}$ from $\sigma_{i_{k-2}}$

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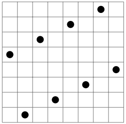

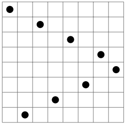

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Positions of σ_{i_k} and σ_{j_k} are fixed for all k as σ_{i_k} does not separate $\sigma_{i_{k-1}}$ from $\sigma_{i_{k-2}}$

$\Rightarrow$ σ is either a parallel alternation or a wedge alternation

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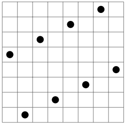

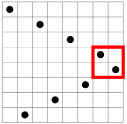

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Positions of σ_{i_k} and σ_{j_k} are fixed for all k as σ_{i_k} does not separate $\sigma_{i_{k-1}}$ from $\sigma_{i_{k-2}}$

$\Rightarrow$ σ is either a parallel alternation or a wedge alternation thus is exceptional or not simple, contradiction.

Proof (case 3)

- σ_i belongs to an interval $I = \{i, i_1\}$ of $\sigma \setminus \sigma_j$ and σ_j belongs to an interval $J = \{j, j_1\}$ of $\sigma \setminus \sigma_i$.

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leads also to a contradiction (almost the same proof)

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leads also to a contradiction (almost the same proof)

$$\sigma \text{ non exceptional, } \underbrace{\pi \preceq \sigma}_{2} \Rightarrow \underbrace{\pi \preceq \tau}_{1}\underbrace{\preceq \sigma}_{1}$$

Constrained patterns (main theorem)

Theorem : $\sigma \neq \pi$ two simple permutations, σ non exceptional.
$\pi \preceq \sigma$ and $|\pi| \geq 3 \Rightarrow \exists$ a simple permutation τ such that
$\pi \preceq \tau \preceq \sigma$ and $|\tau| = |\sigma| - 1$.

$$\sigma \text{ non exceptional, } \underbrace{\pi \preceq \sigma}_{\geq 1} \Rightarrow \pi \preceq \underbrace{\tau \preceq \sigma}_{1}$$

π, σ and τ simple permutations

Proposition : $\underbrace{\pi \preceq \sigma}_{\geq 2} \Rightarrow \underbrace{\pi \preceq \tau}_{2} \preceq \sigma$

Proposition : σ non exceptional, $\underbrace{\pi \preceq \sigma}_{2} \Rightarrow \underbrace{\pi \preceq \tau}_{1} \underbrace{\preceq \sigma}_{1}$

Theorem : σ non exceptional, $\underbrace{\pi \preceq \sigma}_{\geq 1} \Rightarrow \pi \preceq \underbrace{\tau \preceq \sigma}_{1}$

$\pi \qquad\qquad\qquad \preceq \sigma$

π, σ and τ simple permutations

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Theorem : σ non exceptional, $\underbrace{\pi \preceq \sigma}_{\geq 1} \Rightarrow \pi \preceq \underbrace{\tau \preceq \sigma}_{1}$

$\pi \underbrace{\preceq \tau_1}_{2} \qquad \preceq \sigma$

Constrained patterns

π, σ and τ simple permutations

Proposition : $\underbrace{\pi \preceq \sigma}_{\geq 2} \Rightarrow \underbrace{\pi \preceq \tau}_{2} \preceq \sigma$

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Theorem : σ non exceptional, $\underbrace{\pi \preceq \sigma}_{\geq 1} \Rightarrow \pi \preceq \underbrace{\tau \preceq \sigma}_{1}$

$\pi \underbrace{\preceq \tau_1}_{2} \underbrace{\preceq \tau_2}_{2} \quad \preceq \sigma$

Constrained patterns

π, σ and τ simple permutations

Proposition : $\underbrace{\pi \preceq \sigma}_{\geq 2} \Rightarrow \underbrace{\pi \preceq \tau}_{2} \preceq \sigma$

Proposition : σ non exceptional, $\underbrace{\pi \preceq \sigma}_{2} \Rightarrow \underbrace{\pi \preceq \tau}_{1}\underbrace{\preceq \sigma}_{1}$

Theorem : σ non exceptional, $\underbrace{\pi \preceq \sigma}_{\geq 1} \Rightarrow \pi \preceq \underbrace{\tau \preceq \sigma}_{1}$

$\pi \underbrace{\preceq \tau_1}_{2} \underbrace{\preceq \tau_2}_{2} \cdots \underbrace{\preceq \tau_k}_{2} \underbrace{\preceq \sigma}_{1 \textbf{ or } 2}$

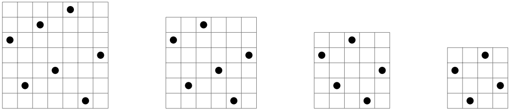
Chains (1)

Theorem : $\pi \neq \sigma$ simple permutations. If $\pi \preceq \sigma$ and $3 \leq |\pi| \Rightarrow \exists$ a chain of simple permutations $\sigma^{(0)} = \sigma, \sigma^{(1)}, \ldots, \sigma^{(k-1)}, \sigma^{(k)} = \pi$ and $m \in \{0 \ldots k\}$ such that $\sigma^{(i)} \preceq \sigma^{(i-1)}$, $|\sigma^{(i-1)}| - |\sigma^{(i)}| = 1$ if $1 \leq i \leq m$, $|\sigma^{(i-1)}| - |\sigma^{(i)}| = 2$ if $m+1 \leq i \leq k$ and if $m < k$ then $\sigma^{(i)}$ is exceptional for $m \leq i \leq k$.

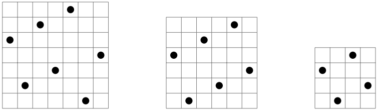
$$\sigma \succ \pi \;\; \Rightarrow$$

$$\sigma = \underbrace{\underbrace{\sigma^{(0)} \succ \sigma^{(1)}}_{1} \succ \cdots \succ \underbrace{\sigma^{(m-1)} \succ}_{1}}_{\textbf{non exceptional}} \underbrace{\underbrace{\sigma^{(m)} \succ}_{2} \cdots \succ \underbrace{\sigma^{(k-1)} \succ \sigma^{(k)}}_{2}}_{\textbf{exceptional}} = \pi$$

Chains : example

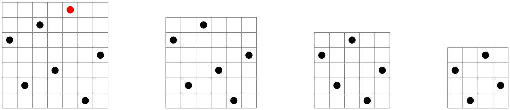


Maximal chain of length 3 from $\sigma = 5263714$ to $\pi = 3142$.



Maximal chain of length 2 from $\sigma = 5263714$ to $\pi = 3142$.

Chains : example



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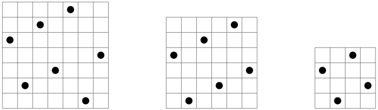

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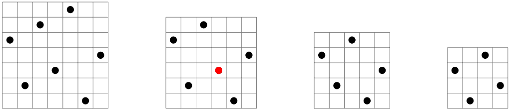

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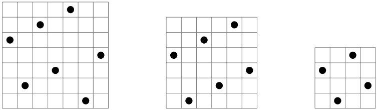

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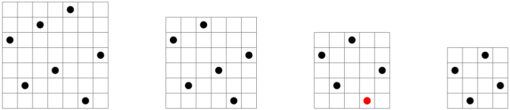

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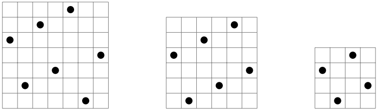

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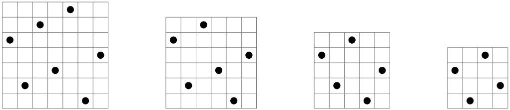

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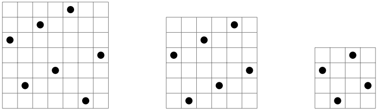

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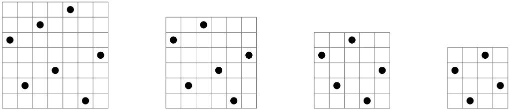

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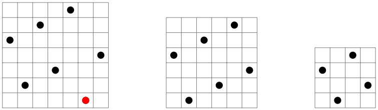

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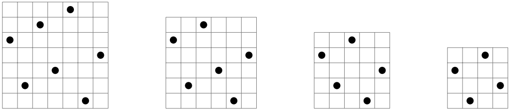

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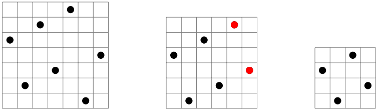

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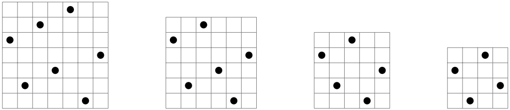

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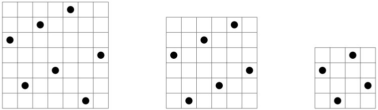

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Chains (2)

Theorem : $\sigma \neq \pi$ two simple permutations, σ non exceptional and $\ell = |\sigma| - |\pi|$. $\pi \preceq \sigma$ and $|\pi| \geq 3 \Rightarrow \exists$ a chain of simple permutations $\sigma^{(0)} = \sigma, \sigma^{(1)}, \ldots, \sigma^{(\ell-1)}, \sigma^{(\ell)} = \pi$ such that $\forall i$, $\sigma^{(i)} \preceq \sigma^{(i-1)}$ and $|\sigma^{(i-1)}| - |\sigma^{(i)}| = 1$.

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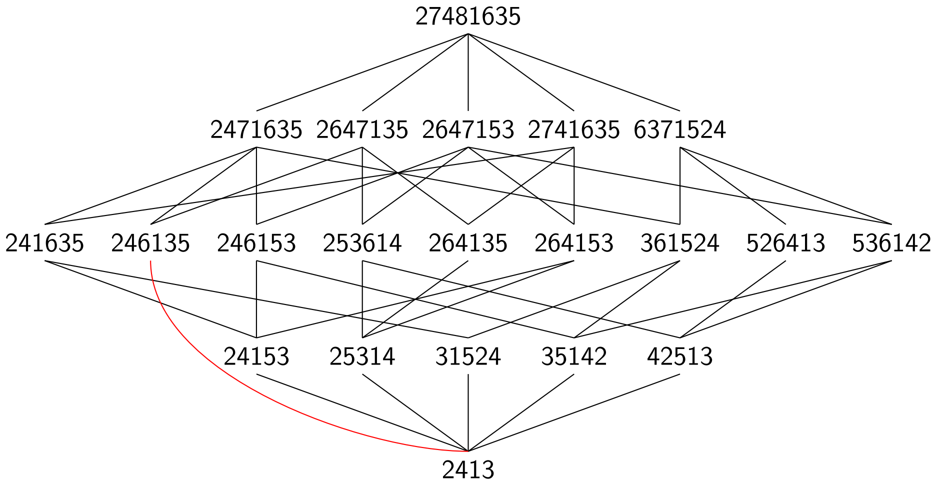
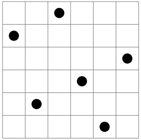
Chains from 27481635 to 2413



Number of parents

Definition : π a simple permutation. We set :
$p_\pi = \# \{\sigma \mid \sigma \text{ is simple}, \pi \preceq \sigma \text{ and } |\sigma| = |\pi| + 1\}$

Proposition : π a simple permutation of size n. Then
$p_\pi = (n+1)(n-3)$.



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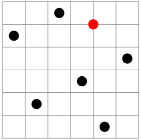

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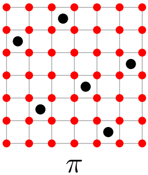

$(n+1)^2$ ways to add a point

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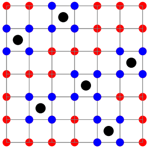

π

$(n+1)^2$ ways to add a point
$4n$ lead to a permutation with an interval of size 2

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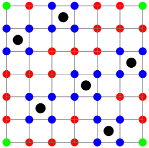

$(n+1)^2$ ways to add a point
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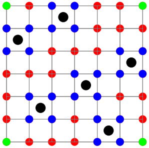

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Then τ_i belongs to an interval of size 2 of τ or is in a corner of the graphical representation of τ.

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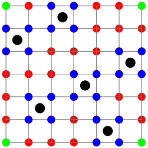

π

$(n+1)^2$ ways to add a point
$4n$ lead to a permutation with an interval of size 2
4 lead to a permutation with an interval of size n
$\Rightarrow p_\pi = (n+1)^2 - 4(n+1) = (n+1)(n-3)$

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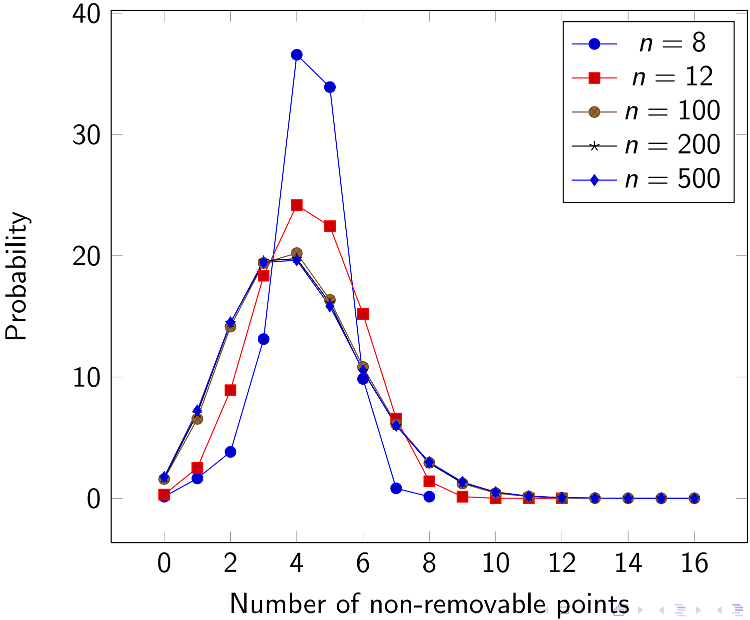
Number of children

Proposition : Let c_n be the average number of children for simple permutations of size n. Then $c_n = n - 4 - \frac{4}{n} + O(\frac{1}{n^2})$.

Proof : Let s_n be the number of simple permutations of size n and e_n be the number of edges between simple permutations of size n and $n-1$. Then $c_n = \frac{e_n}{s_n} = p_{n-1}\frac{s_{n-1}}{s_n} = n(n-4)\frac{s_{n-1}}{s_n}$ and we know that $s_n = \frac{n!}{e^2}\left(1 - \frac{4}{n} + \frac{2}{n(n-1)} + O(n^{-3})\right)$.

Numerical results



Algorithm : Let $B = \{\pi_1 \dots \pi_m\}$ and $C = Av(B)$ a wreath-closed class (i.e. π_i is simple $\forall i$). Then we can recursively compute the set Si_n of simple permutations of size n in C from Si_{n-1} and Si_{n-2} as follow :

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- $\forall \tau \in Si_{n-1}$, $\forall \sigma$ simple permutation obtained from τ by adding a point, if $\sigma \notin B$ and each simple pattern of σ of size $n-1$ is in Si_{n-1}, we add σ to Si_n.
- If n is even, for i from 1 to 4, let σ be the exceptional permutation of type i of size n. If $\sigma \notin B$ and the exceptional permutation of type i of size $n-2$ is in Si_{n-2}, we add σ to Si_n.

Algorithm : proof

Recall : σ non exceptional, $\underbrace{\pi \preceq \sigma}_{\geq 1} \Rightarrow \pi \preceq \underbrace{\tau \preceq \sigma}_{1}$

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Suppose $\sigma \in C$. Then $\exists \tau \prec \sigma$, τ simple of size $n-1$
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Reciprocally, if $\sigma \notin C$, then $\exists i$, $\pi_i \preceq \sigma$ and $\exists \tau$ simple of size $n-1$, $\pi_i \preceq \tau \preceq \sigma$ so $\tau \notin Si_{n-1}$ and we don't add σ to Si_n.

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Almost the same reasoning if σ is exceptional.

Conclusion

- Our goal was to find an algorithm to compute effectively the set of simple permutations in a class. We have it for wreath-closed classes. In other classes ?

- Simple permutation poset $\Rightarrow$ many results interesting in themselves.

Thank you for your attention