# An Asymptotic Version of a Theorem of Knuth

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# Symmetry

# Schensted pairs

$s(d, N) =$ no. of Schensted pairs on partitions $\lambda \vdash N,\ \ell(\lambda) \leq d$

$$s(3, 9) = 94\ 359$$

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 8 | | |
| 9 | | | | |

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 |
| 2 | 4 | 6 | | |
| 8 | | | | |

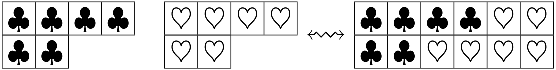
$$\boxed{s(d, N) = \ ?}$$

# Knuth's formula

## Theorem

$$\boxed{s(2, N) = \dim R(2, N)}$$

## Proof.



$\square$

## Corollary

$$s(2, N) = \frac{1}{N+1}\binom{2N}{N}.$$

# A general formula

$$s(d,N) = \sum_{\substack{\lambda \vdash N \\ \ell(\lambda) \le d}} (\dim \lambda)^2,$$

where

$$\dim \lambda = \frac{N!}{\prod_{i=1}^{d} (\lambda_i - i + d)!} \prod_{1 \le i < j \le d} (\lambda_i - \lambda_j + j - i).$$

Challenge: use this formula to estimate

$$s(3, 10^{10}).$$

# Regev's formula

**Theorem**

*For any fixed* $d \geq 1$,

$$s(d,N) \sim (2\pi)^{\frac{1-d}{2}} \left( \prod_{i=0}^{d-1} i! \right) d^{2N+\frac{d^2}{2}} (2N)^{\frac{1-d^2}{2}}$$

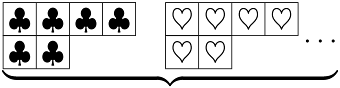
$$= \underbrace{\left( (2\pi)^{\frac{1-d}{2}} \left( \prod_{i=0}^{d-1} i! \right) d^{\frac{d^2}{2}} 2^{\frac{1-d^2}{2}} \right)}_{\text{GUE partition function}} \underbrace{\left( d^{2N} N^{\frac{1-d^2}{2}} \right)}_{\mathbf{S}(d,N) \text{ growth rate}}$$

*as* $N \to \infty$.

# Regev's formula

**Proof.**

"Continuous" Schensted pairs:



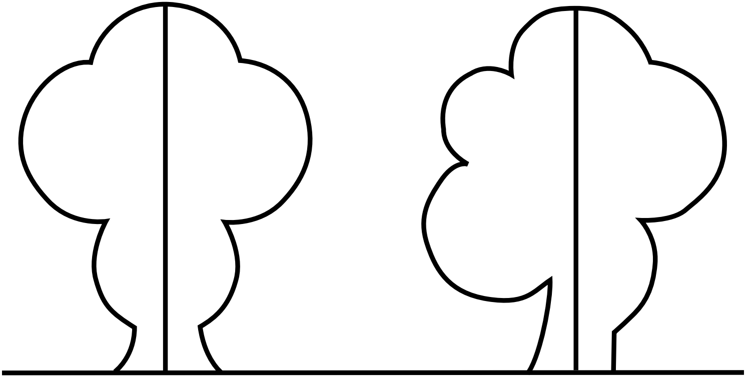
$\beta$ times

$$s(d,N;\beta) := \sum_{\substack{\lambda \vdash N \\ \ell(\lambda)\le d}} (\dim \lambda)^{\beta}$$

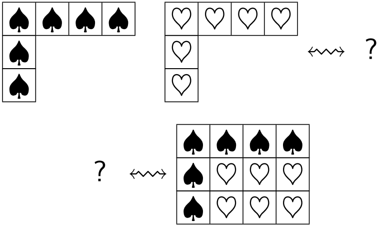
$$C^{\beta}_{d,N}\, s(d,N;\beta) \to \underbrace{\int_{\Omega_{d-1}} e^{-\beta W(y_1,\ldots,y_{d-1})}\, d\mathbf{y}}_{\text{Mehta-Dyson-Selberg}}$$

□

# Asymmetry



# Asymmetry



# Symmetry

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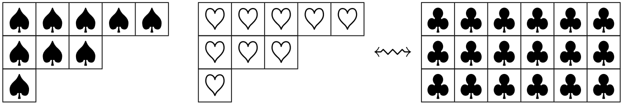
# Asymptotic symmetry



Conjecture:

$$s(d, N) \sim \dim R(d, 2N/d)$$

# Verification of asymptotic symmetry

Exact formula:

$$\dim R(d,q) = \frac{(dq)!}{\prod_{i=0}^{d-1} \frac{(q+i)!}{i!}}$$

Dimension of a $d \times \infty$ strip:

$$\dim R(d,q) \sim (2\pi)^{\frac{1-d}{2}} \left( \prod_{i=0}^{d-1} i! \right) d^{dq+\frac{1}{2}} q^{\frac{1-d^2}{2}}$$

Scaling dictated by symmetry:

$$q \rightsquigarrow 2N/d$$

Reproduces Regev's formula:

$$\dim R(d, 2N/d) \sim (2\pi)^{\frac{1-d}{2}} \left( \prod_{i=0}^{d-1} i! \right) d^{2N+\frac{d^2}{2}} (2N)^{\frac{1-d^2}{2}}$$

# Asymptotic Knuth theorem

**Theorem**

*For any fixed $d \geq 1$,*

$$s(d, dn) \sim \dim R(d, 2n)$$

*as $n \longrightarrow \infty$.*

**Corollary**

*The number of permutations in $\mathbf{S}(dn)$ with no decreasing subsequence of length $d+1$ is asymptotically equal to the number of involutions in $\mathbf{S}(2dn)$ with longest decreasing subsequence of length exactly $d$ and longest increasing subsequence of length exactly $2n$.*

# Error term

Complements: $\mu \subset R(d,q)$,

$$\mu^{*R(d,q)} = (q - \mu_d, q - \mu_{d-1}, \ldots, q - \mu_1)$$

## Theorem

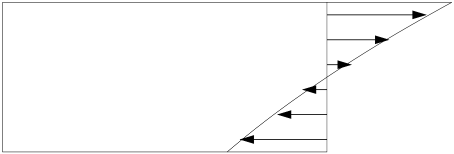
$$s(d, dn) = \dim R(d, 2n) + E(d, dn),$$

*where*

$$E(d, dn) = \underbrace{\frac{1}{2} \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d,2n)}} (\dim \mu - \dim \mu^*)^2}_{\textit{asymmetry}} + \underbrace{\sum_{\substack{\nu \vdash dn \\ \nu_1 > 2n}} (\dim \nu)^2}_{\textit{large deviation}}.$$

# Laplace method

*If you want to understand a sum/integral where the integrand contains a large parameter, the maximum of the integrand is the centre of the universe.*



$$\dim(n + y_1\sqrt{n}, \dots, n + y_d\sqrt{n}) \sim \ ?$$

# Laplace method

## Theorem

*For any fixed* $y_1 > \cdots > y_d,\ y_1 + \cdots + y_d = 0$,

$$\lim_{n\to\infty} C_{d,dn} \dim(n + y_1\sqrt{n}, \ldots, n + y_d\sqrt{n}) = e^{-W(y_1,\ldots,y_d)},$$

*where*

$$W(y_1,\ldots,y_d) = \frac{1}{2}\sum_{i=1}^{d} y_i^2 - \sum_{1\le i<j\le d} \log(y_i - y_j).$$

## Proof.

$$\dim(n + y_1\sqrt{n}, \ldots, n + y_d\sqrt{n})$$
$$= \frac{\Gamma(dn+1)}{\prod_{i=1}^{d} \Gamma(n + y_i\sqrt{n} + i + d + 1)} \prod_{1\le i<j\le d} ((y_i - y_j)\sqrt{n} + j - i).$$

## Laplace method

$$s(d, N; \beta) = \sum_{\substack{\lambda \vdash N \\ \ell(\lambda) \le d}} (\dim \lambda)^\beta$$

$$\lim_{n\to\infty} C^\beta_{d,dn} s(d, dn; \beta) = \int_{\Omega_{d-1}} e^{-\beta W(y_1,\dots,y_d)} d\mathbf{y}$$

$$\Omega_{d-1} = \{y_1 > \cdots > y_d,\ y_1 + \cdots + y_d = 0\} \subset \mathbb{R}^{d-1}$$

Regev: evaluate this (difficult) integral.

# Laplace method

$$\dim R(d,2n) = \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d,2n)}} (\dim \mu)(\dim \mu^*).$$

$$t(d, dn; \gamma, \delta) = \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d,2n)}} (\dim \mu)^\gamma (\dim \mu^*)^\delta.$$

Exactly the same argument:

$$\lim_{n\to\infty} C_{d,dn}^{\gamma+\delta} t(d, dn; \gamma, \delta) = \int_{\Omega_{d-1}} e^{-\gamma W(y_1,\ldots,y_d)} e^{-\delta W(-y_d,\ldots,-y_1)} d\mathbf{y}.$$

# Symmetry returns

$$s(d, dn) \sim \dim R(d, 2n)$$

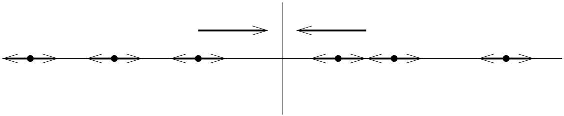
$$\Updownarrow$$

$$\int_{\Omega_{d-1}} e^{-2W(y_1,\ldots,y_d)} d\mathbf{y} = \int_{\Omega_{d-1}} e^{-W(y_1,\ldots,y_d)} e^{-W(-y_d,\ldots,-y_1)} d\mathbf{y}$$

$$\Uparrow$$

$$W(y_1, \ldots, y_d) = W(-y_d, \ldots, -y_1)$$

# Symmetry returns



Energy:

$$W(y_1, \ldots, y_d) = \frac{1}{2} \sum_{i=1}^{d} y_i^2 - \sum_{1 \le i < j \le d} \log(y_i - y_j).$$

Symmetry:

$$\boxed{W(y_1, \ldots, y_d) = W(-y_d, \ldots, -y_1)}$$

# Symmetry returns

$$\boxed{W(y_1, \ldots, y_d) = W(-y_d, \ldots, -y_1)}$$

## Theorem

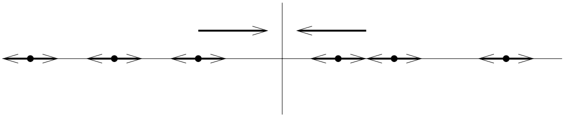
*For any* $0 \leq \gamma < \beta$,

$$\sum_{\substack{\lambda \vdash dn \\ \ell(\lambda) \leq d}} (\dim \lambda)^{\beta} \sim \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d,2n)}} (\dim \mu)^{\gamma} (\dim \mu^*)^{\beta - \gamma}$$

## Corollary

$$\sum_{\substack{\lambda \vdash dn \\ \ell(\lambda) \leq d}} (\dim \lambda)^{2} \sim \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d,2n)}} (\dim \mu)(\dim \mu^*) = \dim R(d, 2n)$$

# Mehta-Dyson integral



Energy:

$$W(t_1,\ldots,t_d) = \frac{1}{2}\sum_{i=1}^{d} t_i^2 - \sum_{1\le i<j\le d} \log(t_i - t_j).$$

Partition function (Mehta-Dyson integral):

$$\Psi(d;\beta) = \int_{\mathcal{W}_d} e^{-\beta W(t_1,\ldots,t_d)} d\mathbf{t}$$

$$\mathcal{W}_d = \{t_1 > \cdots > t_d\} \subset \mathbb{R}^d$$

Mehta-Dyson conjecture:

$$\Psi(d;\beta) = \frac{1}{d!}(2\pi)^{\frac{d}{2}}\beta^{-\frac{d}{2}-\beta\frac{d(d-1)}{4}}\prod_{i=1}^{d}\frac{\Gamma(1+i\frac{\beta}{2})}{\Gamma(1+\frac{\beta}{2})}.$$

- Bombieri: Selberg $\implies$ Mehta-Dyson
- Symmetry $\implies$ $\Psi(d;2)$
- Dyson: $\Psi(d;2k) \implies \Psi(d;\beta)$.
- Symmetry $\implies$ $\Psi(d;2k)$???

## Double-Scaling limit

Baik-Deift-Johansson, Okounkov, Borodin-Okounkov-Olshanski, Johansson:

### Theorem

*For $d, N \to \infty$ at the rate $d \sim 2N^{1/2} + tN^{1/6}$,*

$$s(d, N) \sim F(t)N!,$$

*where*

$$F(t) = \textit{Tracy-Widom distribution function.}$$

$$s(d, dn) = \dim R(d, 2n) + \frac{1}{2} \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d,2n)}} (\dim \mu - \dim \mu^*)^2 + \sum_{\substack{\nu \vdash dn \\ \nu_1 > 2n}} (\dim \nu)^2.$$

Asymptotics of $E(d, dn)$ in double scaling limit???

# Acknowledgements

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