Generating Functions for Wilf Equivalence under Generalized Factor Order

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Outline

A few definitions

Given a poset $\mathcal{P} = (P, \leq_P)$, we define

$$P^* = \{w = w_1 w_2 \ldots w_n \mid n \geq 0 \text{ and } w_i \in P \text{ for all } i\}.$$

And given any $w = w_1 w_2 \ldots w_n \in P^*$,

$$|w| = n \text{ and}$$

$$\sum w = \sum_{i=1}^{n} w_i$$

We will from now on assume that $\mathcal{P} = (\mathbb{N}, \leq)$. (Or at least until the very end.)

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Generalized Factor Order

Given $u, w \in P^*$, we say that $u \leq_{\mathcal{P}} w$ if there is a factor of length $|u|$ of the word w that is componentwise larger than u.

Example

$$231 \leq_{\mathcal{P}} 1424312$$

We say that:

- 243 and 431 are both embeddings of 231 into 1424312.
- the word 231 embeds into 1424312.

A few more definitions

Given any $w \in P^*$,

$$\mathrm{wt}(w) = t^{|w|} x^{\sum w}.$$

Example

$$\mathrm{wt}(2144) = t^4 x^{11}$$

Some sets associated with a word u

Given any $u \in P^*$, define

$$
\begin{aligned}
\mathcal{F}(u) &= \{w \in P^* \mid u \leq_{\mathcal{P}} w\} \\
\mathcal{S}(u) &= \{w \in P^* \mid u \leq_{\mathcal{P}} w \text{ and the last } |u| \text{ characters of } w \text{ is the only} \\
&\qquad \text{embedding of } u \text{ into } w\}, \\
\mathcal{W}(u) &= \{w \in P^* \mid u \leq_{\mathcal{P}} w \text{ and } |w| = |u|\}, \text{ and} \\
\mathcal{A}(u) &= \{w \in P^* \mid u \not\leq_{\mathcal{P}} w\}
\end{aligned}
$$

And the corresponding weight generating functions

$$
\begin{aligned}
F(u;t,x) &= \sum_{w \in \mathcal{F}(u)} \mathrm{wt}(w), \\
S(u;t,x) &= \sum_{w \in \mathcal{S}(u)} \mathrm{wt}(w), \\
W(u;t,x) &= \sum_{w \in \mathcal{W}(u)} \mathrm{wt}(w), \text{ and} \\
A(u;t,x) &= \sum_{w \in \mathcal{A}(u)} \mathrm{wt}(w).
\end{aligned}
$$

Definition (Wilf Equivalence in GFO (Kitaev, Liese, Remmel, Sagan))

$$u \backsim v \Leftrightarrow F(u;t,x) = F(v;t,x)$$

Theorem (Kitaev, Liese, Remmel, Sagan (2009))

$F(u;t,x)$, $S(u;t,x)$, $W(u;t,x)$ and $A(u;t,x)$ are rational.

One can construct a non-deterministic finite automaton for each $u \in P^*$ that recognizes $\mathcal{S}(u)$, implying that $S(u;t,x)$ is rational.

Let $E(t,x)$ be the weight generating function for all words in P^*

$$E(t,x) = \sum_{w \in P^*} \mathrm{wt}(w) = \frac{1}{1 - \sum_{n \geq 1} tx^n} = \frac{1}{1 - tx/(1-x)} = \frac{1-x}{1-x-tx},$$

and therefore

$$\begin{aligned} F(u;t,x) &= S(u;t,x)\frac{1-x}{1-x-tx}, \\ A(u;t,x) &= \frac{1-x}{1-x-tx} - F(u;t,x), \mathit{and} \\ W(u;t,x) &= \frac{t^{|u|}x^{\Sigma(u)}}{(1-x)^{|u|}}. \end{aligned}$$

Some Wilf Equivalences

For words in S_2:

12, 21

For words in S_3:

123, 132, 231, 321
213, 312

For words in S_4:

1234, 1243, 1342, 1432, 2341, 2431, 3421, 4321
1324, 1423, 3241, 4231
2134, 2143, 3412, 4312
3124, 3214, 4123, 4213
2314, 2413, 3142, 4132

The Generating Function $S(u; t, x)$

111	$\frac{t^3x^3}{(1-x)^3}$
112,121,211	$\frac{t^3x^4}{(1-x)^3(1-tx)}$
122,221	$\frac{t^3x^5}{(1-x)^2(1-x-tx+tx^2-t^2x^3)}$
212	$\frac{t^3x^5(1+tx^2)}{(1-x)(1-x+t^2x^3)(1-x-tx+tx^2-t^2x^3)}$
113,131,311	$\frac{t^3x^5}{(1-x)^3(1-tx-tx^2)}$
213,312	$\frac{t^3x^6(1+tx^3)}{(1-x)(1-x+t^2x^4)(1-x-tx+tx^3-t^2x^4)}$
123,132,231,321	$\frac{t^3x^6}{(1-x)^2(1-x-tx+tx^3-t^2x^4)}$
222	$\frac{t^3x^6}{(1-x)(1-2x-tx+x^2+2tx^2-tx^3-t^2x^3+t^2x^4-t^3x^5)}$
133,331	$\frac{t^3x^7}{(1-x)^2(1-x-tx+tx^3-t^2x^4-t^2x^5)}$
313	$\frac{t^3x^7(1+tx^3+tx^4)}{(1-x)(1-x+t^2x^4+t^2x^5)(1-x-tx+tx^3-t^2x^4-t^2x^5)}$
223,232,322	$\frac{t^3x^7}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^5-t^3x^6)}$
323	$\frac{t^3x^8(1+tx^3)}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^5-t^3x^6-t^3x^7+t^3x^8-t^4x^9-t^4x^{10})}$
233,332	$\frac{t^3x^8}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^6-t^3x^7)}$
333	$\frac{t^3x^9}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^6-t^3x^7-t^3x^8)}$

Conjecture (Kitaev, Liese, Remmel, Sagan)

If $u \backsim v$, then v is a rearrangement of u.

Conjecture (Langley, Liese, Remmel)

If $u \backsim v$, then there is a weight preserving bijection $f : P^* \to P^*$ such that for all $w \in P^*$, $f(w)$ is a rearrangement of w and $w \in \mathcal{F}(u) \iff f(w) \in \mathcal{F}(v)$.

Definition

We call such a bijection a rearrangement map that witnesses $u \backsim v$.

A rearrangement map

By computing generating functions, we have seen that

$$123 \backsim 132.$$

We will now illustrate a rearrangement map that witnesses this relation,

$$\Theta : \mathcal{F}(123) \mapsto \mathcal{F}(132).$$

Take a word $w \in \mathcal{F}(123)$, if 132 embeds into w, then the bijection does nothing.

What if 132 does **not** embed into w?

Note: It is impossible for two embeddings of 123 into a word to overlap without having an embedding of 132.

The map is: For each embedding of 123, switch the roles of the 2 and 3 and see if an embedding of 123 is created,

1. if not then you are done.
2. if so continue switching until there is no embedding of 123.

Example

Suppose $w = 3122223131223$.

- 3122223131223

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- 3122232131223

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- 3122322131223

Example

Suppose $w = 3122223131223$.

- 3123222131223

Example

Suppose $w = 3122223131223$.

- 3132222131223

Example

Suppose $w = 3122223131223$.

- 3132222131232

Example

Suppose $w = 3122223131223$.

- 3132222131322

Definition

For any word u, let u_{inc} be the longest weakly increasing prefix of u. If $u = u_{inc}v$ and v is weakly decreasing, then we shall say that u has an *increasing/decreasing factorization* and denote v as u_{dec}.

Example

If $u = 124554431$, then $u_{inc} = 12455$ and $u_{dec} = 4431$.

Some necessary definitions

For the theorem that follows, we define

$$D^{(i)}(u) = \{n - i + j : 1 \leq j \leq i \text{ and } u_j > u_{n-i+j}\}$$

and $d_i(u) = \sum_{n-i+j \in D^{(i)}(u)} (u_j - u_{n-i+j})$.

Example

If $u = 1\ 2\ 3\ 4\ 4\ 3\ 1\ 1$ and $i = 5$, then by considering the diagram

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 4 & 3 & 1 & 1 \\ & & & 1 & 2 & 3 & \underline{4} & \underline{4} \end{array},$$

we see that $D^{(5)}(u) = \{7, 8\}$ and $d_5(u) = (4-1) + (4-1) = 6$.

The main result

Theorem (Langley, Liese, Remmel)

Let $u = u_1 u_2 \dots u_n \in P^*$ *have an increasing/decreasing factorization. For* $1 \leq i \leq n-1$, *let* $s_i = u_{i+1} u_{i+2} \dots u_n$ *and* $d_i = d_i(u)$. *Also let* $s_n = \varepsilon$ *and* $d_n = 0$. *Then*

$$S(u;t,x) = \frac{t^n x^{\Sigma(u)}}{t^n x^{\Sigma(u)} + (1-x-tx)\sum_{i=1}^{n} t^{n-i} x^{d_i + \Sigma(s_i)} (1-x)^{i-1}}.$$

Example

Suppose $u = 1342$, then $s_1 = 342$, $s_2 = 42$, $s_3 = 2$,$s_4 = \varepsilon$ and by convention $d_4 = 0$.

$$\begin{matrix} 1 & 3 & 4 & 2 \\ & 1 & 3 & \underline{4} \end{matrix},$$

Thus, $d_3 = 2$.

$$\begin{matrix} 1 & 3 & 4 & 2 \\ & & 1 & \underline{3} \end{matrix},$$

Thus, $d_2 = 1$.

$$\begin{matrix} 1 & 3 & 4 & 2 \\ & & & 1 \end{matrix},$$

Thus, $d_1 = 0$. Using the Theorem, we obtain $S(1342; t, x) =$

$$\frac{t^4x^{10}}{t^4x^{10} + (1 - x - tx)(t^3x^9 + t^2x^7(1 - x) + tx^4(1 - x)^2 + (1 - x)^3)}.$$

Conjecture (Langley, Liese, Remmel)

For $u \in P^*$, $S(u;t,x) = \frac{x^s t^r}{P(u;t,x)}$ where $P(u;t,x)$ is a polynomial if and only if u has an increasing/decreasing factorization.

And now a sketch of the proof of the main theorem.

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And now a sketch of the proof of the main theorem.

A sketch of the proof

Why isn't it the case that

$$S(u;t,x) = A(u;t,x)W(u;t,x)?$$

Suppose $u = 1243$, take $a = 1211332 \in \mathcal{A}(u)$ and $w = 4444 \in \mathcal{W}(u)$.

Notice that

$$12113324444 \notin \mathcal{S}(u).$$

This is because we have embeddings that overlap in a and w. Namely, 3244 and 2444.

For each $1 \le i \le n-1$, we define $\mathcal{S}^{(i)}(u)$ to be set of all words $w = w_1 \dots w_m$ such that

- $u \le w_{m-n+1} \dots w_m$ (so that u embeds into the suffix of length n of w) and
- the left-most embedding of u into w starts at position $m - 2n + i + 1$.

We then let

$$S^{(i)}(u;t,x) = \sum_{w \in \mathcal{S}^{(i)}(u)} \mathrm{wt}(w) = \sum_{w \in \mathcal{S}^{(i)}(u)} x^{\Sigma(w)} t^{|w|}.$$

Thus

$$S(u;t,x) = A(u;t,x)W(u;t,x) - \bigcup_{i=1}^{n-1} \mathcal{S}^{(i)}(u;t,x). \quad (1)$$

Lemma

Let $u = u_1 u_2 \dots u_n \in P^*$ have an increasing/decreasing factorization. Then for $1 \le i \le n-1$,

$$S^{(i)}(u;t,x) = S(u;t,x) t^{n-i} x^{d_i + \Sigma(s_i)} \left(\frac{1}{1-x} \right)^{n-i}.$$

Consider $u = 1\,2\,6\,5\,3\,2$, so that $u_{inc} = 1\,2\,6$ and $u_{dec} = 5\,3\,2$, and let $i = 4$. Let's create a word v in $\mathcal{S}^{(i)}(u)$.

$$\begin{array}{cccccccccccc}
v = & (\cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \star & \star) & _ & _ & , \\
 & & & 1 & 2 & 6 & 5 & 3 & 2 & & & \\
 & & & & & 1 & 2 & 6 & 5 & 3 & 2 &
\end{array}$$

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Let $u = u_1 u_2 \ldots u_n \in P^*$ *have an increasing/decreasing factorization. Then for* $1 \leq i \leq n-1$*,*

$$S^{(i)}(u;t,x) = S(u;t,x)t^{n-i}x^{d_i+\Sigma(s_i)}\left(\frac{1}{1-x}\right)^{n-i}.$$

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\end{array},$$

Theorem (Langley, Liese, Remmel)

If $u, v \in P^*$ have increasing/decreasing factorizations, then $u \backsim v$ if and only if u is a rearrangement of v.

Lemma

Suppose $u = u_1 \dots u_n$ is a rearrangement of $v = v_1 \dots v_n$ and that u and v have increasing/decreasing factorizations. Then for all $1 \leq i \leq n-1$,

$$d_i(u) + \Sigma(s_i(u)) = d_i(v) + \Sigma(s_i(v)).$$

This lemma verifies $\Leftarrow$.

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$$d_i(u) + \Sigma(s_i(u)) = d_i(v) + \Sigma(s_i(v)).$$

This lemma verifies $\Leftarrow$.

To prove $\Rightarrow$, suppose $u \backsim v$. WLOG, let $u = u_1 u_2 \ldots u_n$ and $v = v_1 v_2 \ldots v_n$ be nondecreasing.

First note that $u \backsim v$ implies $\Sigma(u) = \Sigma(v)$ and $|u| = |v|$, so the numerators of $S(u;t,x) = S(v;t,x)$ are equal.

Equating denominators,

$$t^n x^{\Sigma(v)} + (1-x-tx) \sum_{i=1}^{n} t^{n-i} x^{\sum_{j=i+1}^{n} v_j} (1-x)^{i-1}$$
$$= t^n x^{\Sigma(u)} + (1-x-tx) \sum_{i=1}^{n} t^{n-i} x^{\sum_{j=i+1}^{n} u_j} (1-x)^{i-1}.$$

Simplifying,

$$\sum_{i=1}^{n} t^{n-i} x^{\sum_{j=i+1}^{n} v_j} (1-x)^{i-1} = \sum_{i=1}^{n} t^{n-i} x^{\sum_{j=i+1}^{n} u_j} (1-x)^{i-1}.$$

Hence for each i, $1 \leq i \leq n$, we have

$$x^{\sum_{j=i+1}^{n} v_j} = x^{\sum_{j=i+1}^{n} u_j},$$

and therefore $u = v$.

Theorem (Langley, Liese, Remmel)

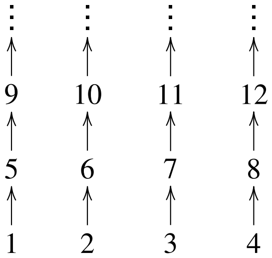
Wilf equivalence partitions words of length 3 into the following 6 equivalence classes.

1. $\{aaa\}$
2. $\{aab, aba, baa\}$ if $a < b$
3. $\{aab, baa\}$ and $\{aba\}$ if $a > b$
4. $\{bac, cab\}$ and $\{abc, acb, cba, bca\}$ if $a < b < c$.

The mod k ordering: We can use the poset $\mathcal{P}_k = \{\mathbb{N}, \leq_k\}$, where for $m, n \in \mathbb{N}$,

$$m \leq_k n \Leftrightarrow m \leq n \text{ and } m \equiv n (\text{mod k}).$$

The Hasse diagram for the mod 4 ordering looks like:



Example

In the mod 3 order, 1111 does not embed into 1435, but 1132 does.

Definition

A word $u = u_1 \ldots u_n$ has the **mod k-nonoverlapping property** if there is no i, $1 \leq i \leq n-1$, such that $u_{n-i+j} \equiv u_j \mod k$ for all $j = 1, \ldots, i$.

Example

Suppose $k = 3$, then $u = 1\ 2\ 2$ has the mod 3-nonoverlapping property.

Example

Also, any permutation of $\{1, 2, \ldots, k\}$ has the mod k-nonoverlapping property.

Suppose $u = u_1 \ldots u_n$ has the mod k-nonoverlapping property.

In this case,

$$\mathcal{S}_k(u) = \mathcal{A}_k(u)\mathcal{W}_k(u)$$

and

$$\begin{aligned} S_k(u;t,x) &= A_k(u;t,x)W_k(u;t,x) \\ &= \frac{(1-x)}{(1-x-tx)}(1 - S_k(u;t,x))\frac{t^n x^{\Sigma(u)}}{(1-x^k)^n}. \end{aligned}$$

Thus,

$$S_k(u;t,x) = \frac{t^n x^{\Sigma(u)}}{t^n x^{\Sigma(u)} + [k]_x(1-x-tx)(1-x^k)^{n-1}}$$

where $[k]_x = \frac{1-x^k}{1-x} = 1 + x + \cdots + x^{k-1}$.

Note that when using $\mathcal{P}_k$, it is not the case that $u \backsim_k v$ implies u and v are rearrangements.

Example

$1\ 4\ 2 \backsim_k 2\ 2\ 3$ for all $k \geq 2$.

Example

$1\ 4 \backsim_k 2\ 3$ for all $k \geq 4$.

Definition (A generalization of increasing/decreasing factorization)

A word $u = u_1 \ldots u_n \in P^*$ has the **mod k-comparison condition** if whenever $i, s \in \mathcal{C}_k(u)$, $n - i < s$, and $u_j > u_{n-i+j}$ where $1 \leq j \leq i$, then $u_{n-s+n-i+j} \leq u_{n-i+j}$ if $n - s + n - i + j \leq n$.

Theorem (Langley, Liese, Remmel)

If $k \geq 2$ and $u = u_1 \ldots u_n \in P^*$ has the mod k-comparison condition, then $S_k(u; t, x) =$

$$\frac{t^n x^{\Sigma(u)}}{t^n x^{\Sigma(u)} + [k]_x (1 - x - tx)((1 - x^k)^{n-1} + \sum_{i \in \mathcal{C}_k(u)} t^{n-i} x^{d_{i,k} + \Sigma(s_i)} (1 - x^k)^{i-1})}.$$

Thank you!

Questions?