Crossings and patterns in signed permutations

Sylvie Corteel, Matthieu Josuat-Vergès, Jang-Soo Kim

Université Paris-sud 11, Université Paris 7

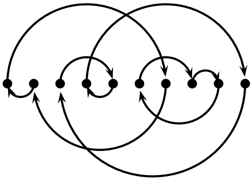
Permutation Patterns

Introduction

A crossing of a permutation σ is a couple (i, j) such that $i < j \leq \sigma(i) < \sigma(j)$, or $\sigma(i) < \sigma(j) < i < j$.

Example

$\sigma = 715\ 10\ 482963$



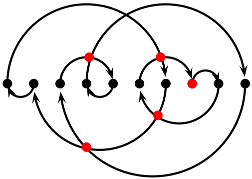
The crossings and 13-2 are equidistributed in permutations.
This is also the same as *superfluous ones* in permutation tableaux.
[Corteel Nadeau, Steingrímsson Williams]

Introduction

A crossing of a permutation σ is a couple (i, j) such that $i < j \leq \sigma(i) < \sigma(j)$, or $\sigma(i) < \sigma(j) < i < j$.

Example

$\sigma = 715\ 10\ 482963$



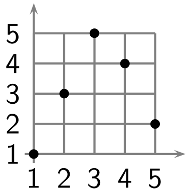
The crossings and 13-2 are equidistributed in permutations.
This is also the same as *superfluous ones* in permutation tableaux.
[Corteel Nadeau, Steingrímsson Williams]

Introduction

Definition

An occurrence of the pattern 13-2 in $\sigma \in \mathfrak{S}_n$ is a triple $(i, i+1, j)$ such that $\sigma(i) < \sigma(j) < \sigma(i+1)$.

Example

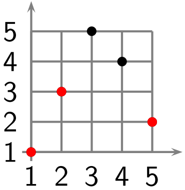


Introduction

Definition

An occurrence of the pattern 13-2 in $\sigma \in \mathfrak{S}_n$ is a triple $(i, i+1, j)$ such that $\sigma(i) < \sigma(j) < \sigma(i+1)$.

Example

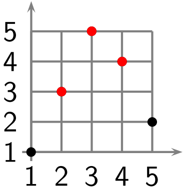


Introduction

Definition

An occurrence of the pattern 13-2 in $\sigma \in \mathfrak{S}_n$ is a triple $(i, i+1, j)$ such that $\sigma(i) < \sigma(j) < \sigma(i+1)$.

Example



Introduction

Definition

A *permutation tableau* is a Young diagram filled with 0's and 1's, such that:

- There is at least a 1 per column,
- The pattern $\begin{array}{ccc} & & 1 \\ & & \vdots \\ 1 & \dots & 0 \end{array}$ is forbidden.

Example

1	0	0	1
1	1	0	1
0	0	1	
1	1		

Introduction

Type B permutation tableaux: defined by Lam and Williams (in relation with geometric objects such as orthogonal grassmannian...)

These are roughly conjugate-symmetric permutation tableaux, and are in bijection with signed permutations.

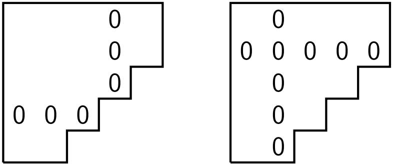
Question: are there some notions of crossings and patterns for signed permutations ?

Type B permutation tableaux

Remark: A conjugate-symmetric permutation tableau contains no zero-row.

Definition

A type B permutation tableau is obtained from a conjugate-symmetric permutation tableau by adding some zero-rows and zero-columns the following way:

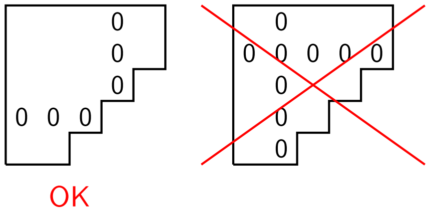


Type B permutation tableaux

Remark: A conjugate-symmetric permutation tableau contains no zero-row.

Definition

A type B permutation tableau is obtained from a conjugate-symmetric permutation tableau by adding some zero-rows and zero-columns the following way:

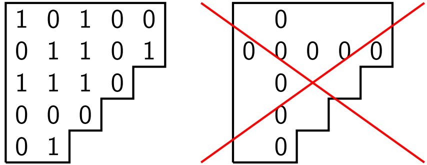


Type B permutation tableaux

Remark: A conjugate-symmetric permutation tableau contains no zero-row.

Definition

A type B permutation tableau is obtained from a conjugate-symmetric permutation tableau by adding some zero-rows and zero-columns the following way:

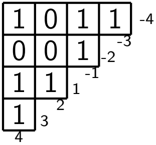


OK

The zig-zag bijection

We use a bijection of [Steingrímsson Williams]. Label the boundary of the permutation tableau with integers for $-n$ to n. The image of i is obtained by taking a zig-zag path, the direction East or South changing at each 1.

Example

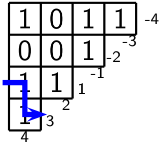


$$\pi = 3, 1, 4, -2.$$

The zig-zag bijection

We use a bijection of [Steingrímsson Williams]. Label the boundary of the permutation tableau with integers for $-n$ to n. The image of i is obtained by taking a zig-zag path, the direction East or South changing at each 1.

Example

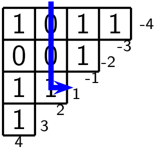


$\pi = 3, 1, 4, -2.$

The zig-zag bijection

We use a bijection of [Steingrímsson Williams]. Label the boundary of the permutation tableau with integers for $-n$ to n. The image of i is obtained by taking a zig-zag path, the direction East or South changing at each 1.

Example

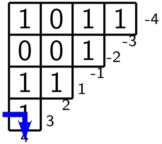


$\pi = 3, 1, 4, -2.$

The zig-zag bijection

We use a bijection of [Steingrímsson Williams]. Label the boundary of the permutation tableau with integers for $-n$ to n. The image of i is obtained by taking a zig-zag path, the direction East or South changing at each 1.

Example

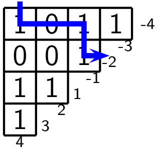


$\pi = 3, 1, 4, -2.$

The zig-zag bijection

We use a bijection of [Steingrímsson Williams]. Label the boundary of the permutation tableau with integers for $-n$ to n. The image of i is obtained by taking a zig-zag path, the direction East or South changing at each 1.

Example



$$\pi = 3, 1, 4, -2.$$

Crossings for signed permutations

Definition

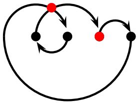
A *crossing* of a signed permutation is a pair $(i, j) \in [n]^2$ such that

- either $i < j \le \pi(i) < \pi(j)$,
- or $i > j > \pi(i) > \pi(j)$,
- or $-i < j \le -\pi(i) < \pi(j)$.

We use an arrow notation such that this corresponds to proper intersection between arrows, or the limit case of two arrows 

Example

$$\pi = 3, 1, 4, -2.$$

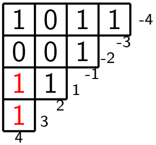


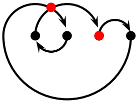
Theorem

Via the zig-zag bijection,

- *the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,*
- $i > 0$ *is such that* $\pi(i) \geq i$ *iff* i *label a vertical step in the South-East boundary of the permutation tableau,*
- *the number of* $i > 0$ *with* $\pi(i) < 0$ *is the number of 1's in the diagonal of the permutation tableau.*

Example





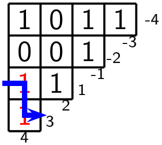
$\pi = 3, 1, 4, -2.$

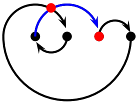
Theorem

Via the zig-zag bijection,

- *the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,*
- *$i > 0$ is such that $\pi(i) \geq i$ iff i label a vertical step in the South-East boundary of the permutation tableau,*
- *the number of $i > 0$ with $\pi(i) < 0$ is the number of 1's in the diagonal of the permutation tableau.*

Example





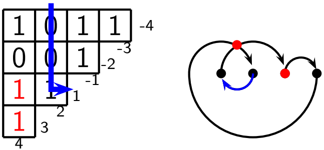
$\pi = 3, 1, 4, -2.$

Theorem

Via the zig-zag bijection,

- *the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,*
- $i > 0$ *is such that* $\pi(i) \geq i$ *iff* i *label a vertical step in the South-East boundary of the permutation tableau,*
- *the number of* $i > 0$ *with* $\pi(i) < 0$ *is the number of 1's in the diagonal of the permutation tableau.*

Example



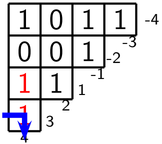
$\pi = 3, 1, 4, -2.$

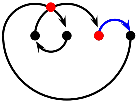
Theorem

Via the zig-zag bijection,

- *the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,*
- *$i > 0$ is such that $\pi(i) \geq i$ iff i label a vertical step in the South-East boundary of the permutation tableau,*
- *the number of $i > 0$ with $\pi(i) < 0$ is the number of 1's in the diagonal of the permutation tableau.*

Example





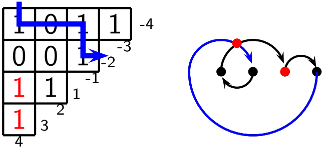
$$\pi = 3, 1, 4, -2.$$

Theorem

Via the zig-zag bijection,

- *the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,*
- *$i > 0$ is such that $\pi(i) \geq i$ iff i label a vertical step in the South-East boundary of the permutation tableau,*
- *the number of $i > 0$ with $\pi(i) < 0$ is the number of 1's in the diagonal of the permutation tableau.*

Example



$\pi = 3, 1, 4, -2.$

Eulerian numbers of type B

There are some definitions of ascents and exceedances in signed permutations [Brenti, Chow] whose distribution are type B Eulerian numbers. In our context, it is interesting to define:

$$\operatorname{twex}(\pi) = \#\{\, i \mid \pi(i) \geq i\} + \lfloor \frac{\operatorname{neg}(\pi)}{2} \rfloor$$

Theorem

Let

$$B_{n,k}(q) = \sum_{\pi \text{ with } \operatorname{twex}(\pi)=k} q^{\operatorname{cr}(\pi)},$$

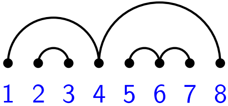
Then $B_{n,k}(q)$ is a q-analog of type B Eulerian numbers such that $B_{n,k}(q) = B_{n,n-k}(q)$.

Non-crossing partitions

A set partition is non-crossing if there are no $i < j < k < \ell$ with i, j in a same block, k, ℓ a one other block.

$$\pi = \{\{1,4,8\},\{2,3\},\{5,6,7\}\}$$

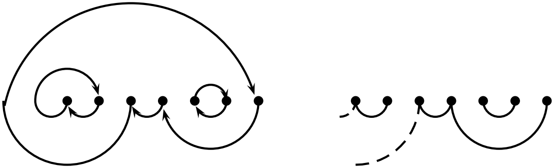
There is a bijection between non-crossing permutations and non-crossing partitions given by the cycle decomposition.



Similarly, non-crossing signed permutations are in bijection with *non-crossing partitions of type B*.

Non-crossing partition of classical types are defined as a sublattice of a Coxeter group. Combinatorial description in type B: a type B non-crossing partition is a couple a (type A) non-crossing partition, and a subset of the non-nested blocks.

There is a bijection with signed permutations having no crossing, for example with $\pi = -2, 1, -7, 3, 6, 5, 4$:



We have $B_{n,k}(0) = \binom{n}{k}^2$, the Narayana number of type B.

A pattern for signed permutation ?

There is a definition of "31-2" pattern for signed permutation such that the distribution is the same as crossings, and an associated notion of signed ascents such that 31-2 gives a q-analog of type B Eulerian numbers.

Definition

- $31\text{-}2(\pi) = \#\{ \ (i,j) \mid$ such that $i < j$, and $|\pi(i)| > |\pi(j)| > |\pi(i+1)|$ or $\pi(i) > -\pi(j) \geq |\pi(i+1) \ \}$
- $\text{pasc}(\pi) = \#\{ \ i \mid \}$

The proof is quite indirect: there is a recursive decomposition of type B permutation tableaux that can be interpreted in terms of weighted Motzkin paths, and then there is a bijection between paths and signed permutations.

Conclusion

- Are there nice enumeration formulas for crossings in signed permutations (case of permutations: [J-V, Corteel, Rubey, Prellberg]) ?
- Is there a better definition of the signed pattern 31-2 ?
- *Snakes* defined by Arnol'd are the signed analog of alternating permutations, are our statistic useful in this context ?

thanks
for your
attention