Composition and partition matrices, bivincular patterns and (2+2)-free posets

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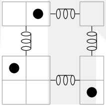
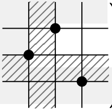
with: Anders Claesson & Martina Kubitzke (RU)

treatment a Pour 2010 Pints

The bivincular pattern $2|3\bar{1}$

Let $S_n(2|3\bar{1})$ be the set of permutations π of $\{1,\ldots,n\}$ such that there do not exist indices $i < k$ satisfying:

$$\pi_i < \pi_{i+1} \text{ and } \pi_i = \pi_k + 1.$$

$S_n(2|3\bar{1}) = S_n\Big($  $\Big) = S_n\Big($  $\Big)$

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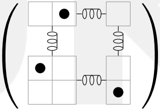
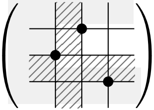
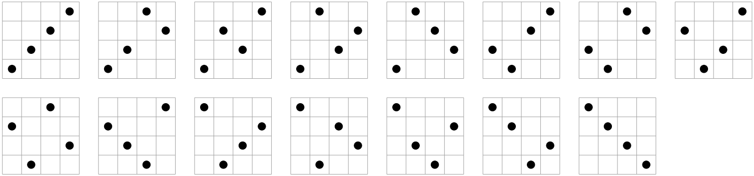
$S_n(2|3\bar{1}) = S_n$  $= S_n$ 

All permutation diagrams of $S_4(2|3\bar{1})$:



Ascent sequences

A sequence $a = (a_1, \ldots, a_n)$ of non-negative integers is called an ascent sequence if $a_1 = 0$ and

$$a_i \in \{0, 1, \ldots, 1 + \operatorname{asc}(a_1, \ldots, a_{i-1})\}$$

for all $1 < i \leq n$. [Note $\operatorname{asc}(\underline{0}, 1, 0, \underline{0}, 2, 0) = 2$.]

Let $\mathcal{A}_n$ be set the of all ascent sequences of length n.

$$\begin{aligned}\mathcal{A}_4 = \{&0000, 0001, 0010, 0011, 0012, 0100, 0101,\\ &0102, 0110, 0111, 0112, 0120, 0121, 0122, 0123.\}\end{aligned}$$

Integer matrices

Let Int_n be the collection of all upper triangular matrices taking values in $\mathbb{N} = \{0, 1, \ldots\}$ such that

- the entries sum to n, and
- there are no rows or columns of 0s.

$$\mathsf{Int}_4 = \Bigg\{ \begin{bmatrix} 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Bigg\}.$$

(2+2)-free posets

A partially ordered set $(P, \leq_P)$ is called $(\mathbf{2}+\mathbf{2})$-free if it contains no induced sub-poset isomorphic to $(\mathbf{2}+\mathbf{2}) =$ 

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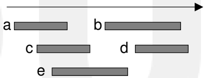
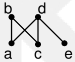
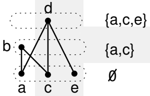
Such posets arise as interval orders:







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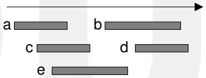

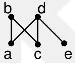

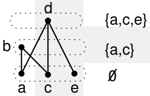

Theorem 1 (Not ours!)

A poset P is $(\mathbf{2}+\mathbf{2})$-free iff the collection of strict order ideals $\{D(x) = \{y < x\} : x \in P\}$ may be linearly ordered by inclusion.

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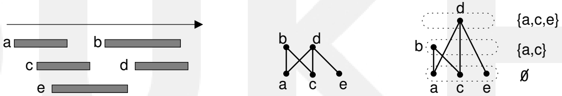


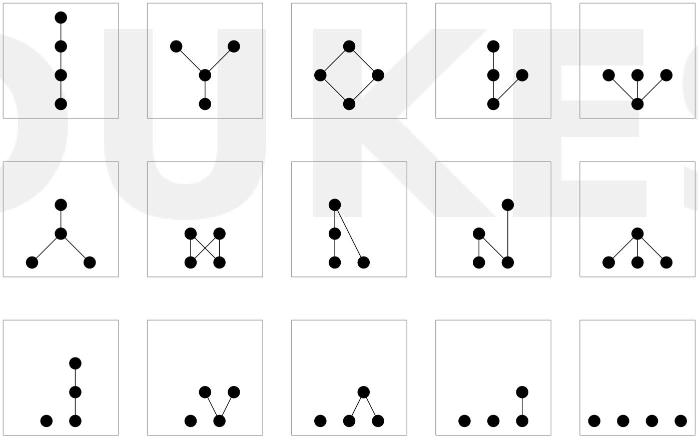
Theorem 1 (Not ours!)

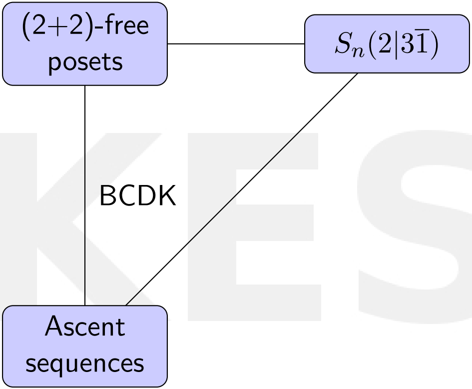
A poset P is $(\mathbf{2}+\mathbf{2})$-free iff the collection of strict order ideals $\{D(x) = \{y < x\} : x \in P\}$ may be linearly ordered by inclusion.

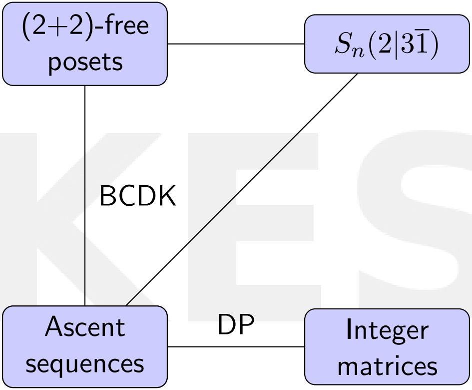
Clearly $D(a) \subseteq D(c) \subseteq D(e) \subseteq D(b) \subseteq D(d)$.

Let $\mathcal{P}_n$ the the collection of all *different* $(\mathbf{2}+\mathbf{2})$-free posets on n elements.

All posets in $\mathcal{P}_4$:




(2+2)-free posets
$S_n(2|3\bar{1})$
BCDK
Ascent sequences


(2+2)-free posets
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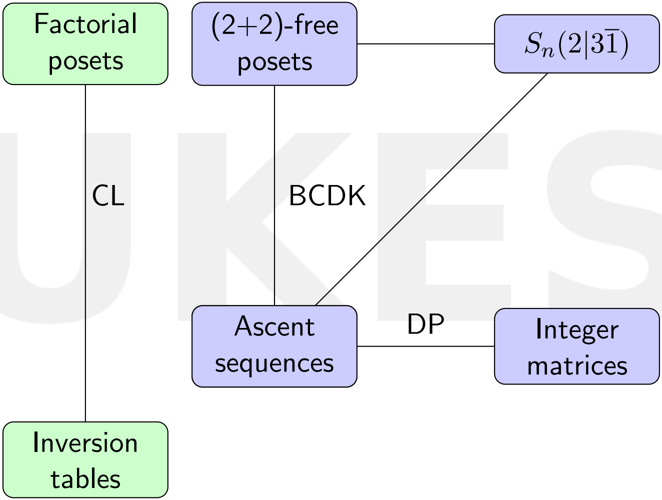
DP
Integer matrices


Factorial posets
(2+2)-free posets
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CL
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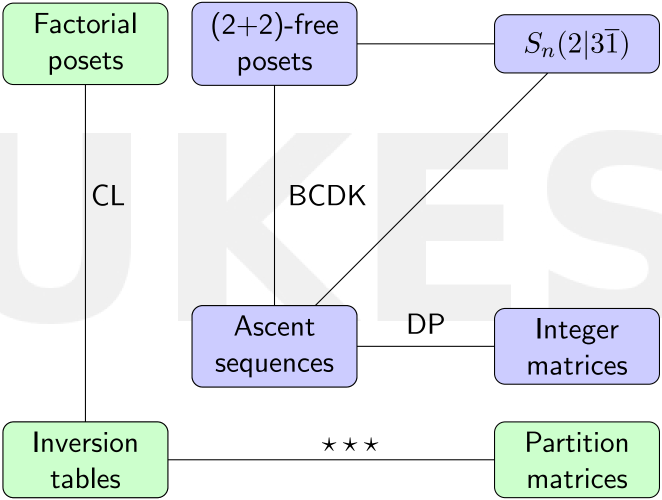
Factorial posets
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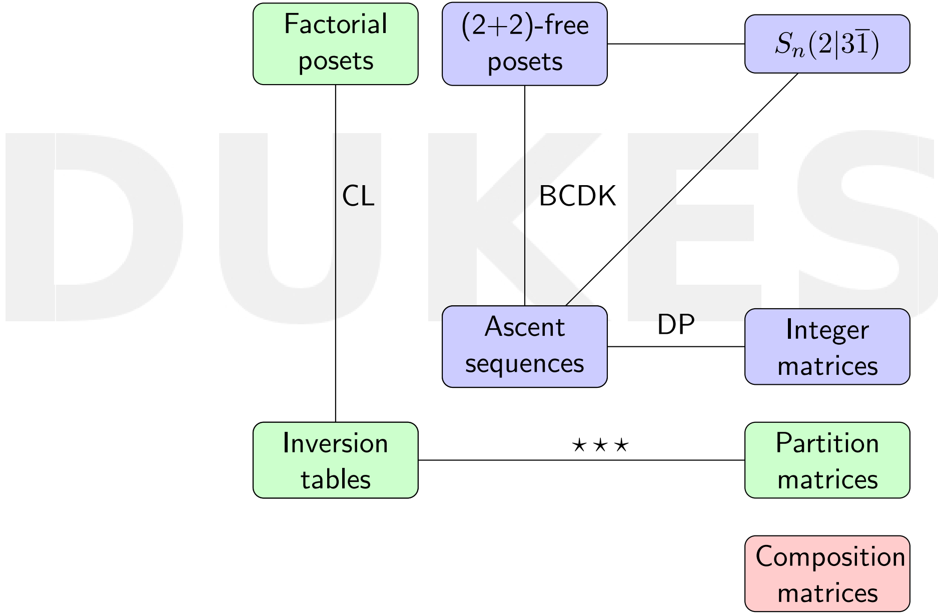
Partition matrices


Factorial posets
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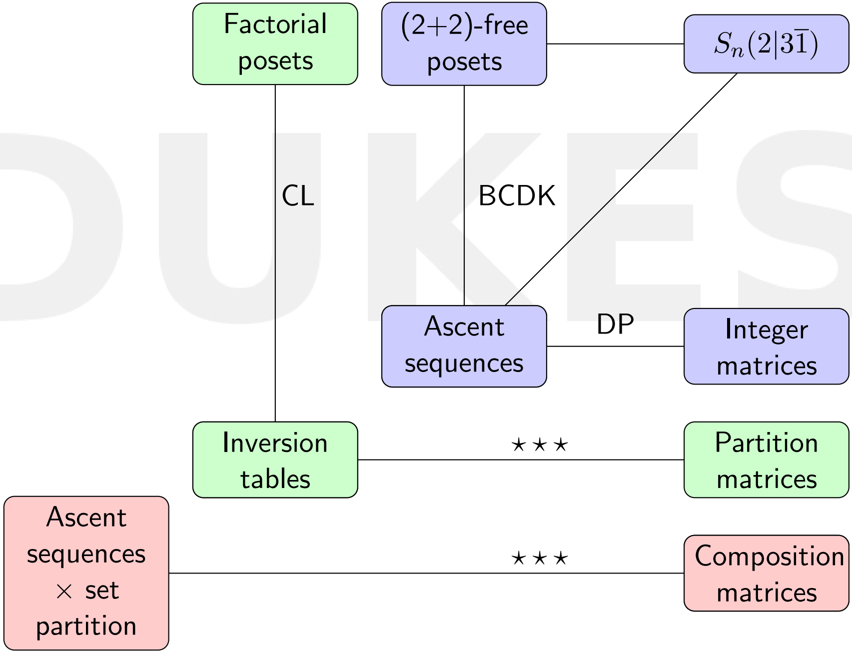
Factorial posets
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Partition matrices
Ascent sequences × set partition
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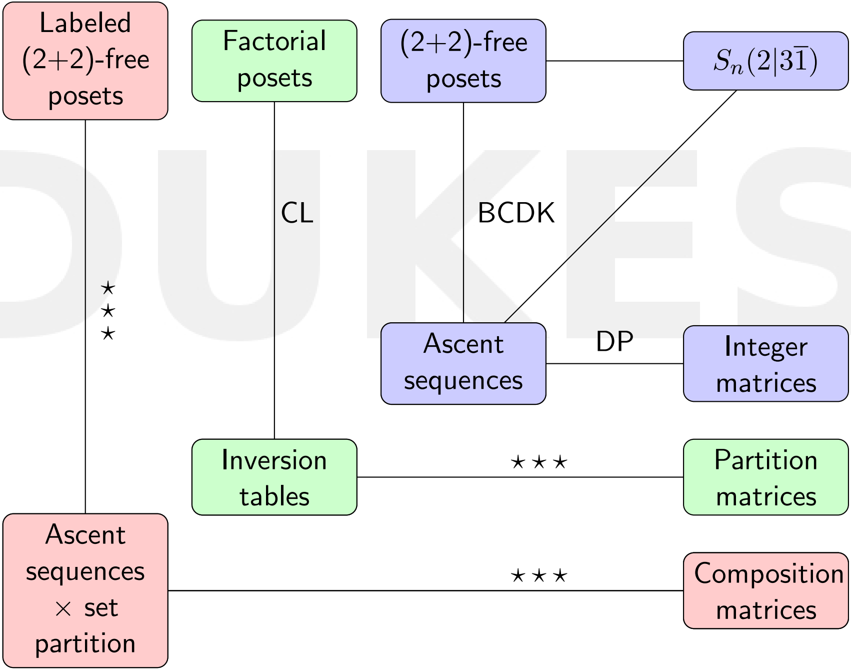
Composition matrices


Labeled (2+2)-free posets
Factorial posets
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Composition matrices

Integer matrices $\mapsto$ ascent sequences

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathsf{Int}_9$$

Integer matrices $\mapsto$ ascent sequences

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Integer matrices $\mapsto$ ascent sequences

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & \color{red}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma(M) = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$$

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Integer matrices $\mapsto$ ascent sequences

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \end{pmatrix}$$

$$\Gamma(M) = (x_1, x_2, x_3, 2, 0, 3, 1, 1, 1)$$

Integer matrices $\mapsto$ ascent sequences

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Gamma(M) = (x_1, x_2, x_3, 2, 0, 3, 1, 1, 1)$$

Integer matrices $\mapsto$ ascent sequences

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

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Integer matrices $\mapsto$ ascent sequences

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Integer matrices $\mapsto$ ascent sequences

$$\begin{pmatrix} 1 \end{pmatrix}$$

$$\Gamma(M) = (x_1, 1, 1, 2, 0, 3, 1, 1, 1)$$

Integer matrices $\mapsto$ ascent sequences

$$\emptyset$$

$$\Gamma(M) = (0, 1, 1, 2, 0, 3, 1, 1, 1)$$

Integer matrices $\mapsto$ ascent sequences

Is it always that easy?!

Integer matrices $\mapsto$ ascent sequences

No!

Integer matrices $\mapsto$ ascent sequences

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Integer matrices $\mapsto$ ascent sequences

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Integer matrices $\mapsto$ ascent sequences

$$\begin{pmatrix} 1 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 1 & 0 & \mathbf{1} & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{2} & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Integer matrices $\mapsto$ ascent sequences

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Integer matrices $\mapsto$ ascent sequences

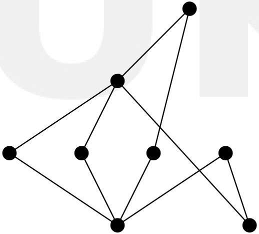
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 1 & 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & 1 & 1 & 1 & \mathbf{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Integer matrices $\mapsto$ ascent sequences

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 1 & 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & 1 & 1 & 1 & \mathbf{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(2+2)-free posets $\mapsto$ ascent sequences

How can one decompose such posets?
There are 3 rules ...



$x = (x_1, \ldots, x_8)$?

(2+2)-free posets ↦ ascent sequences

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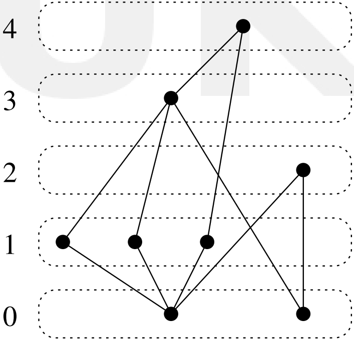

$x = ?$

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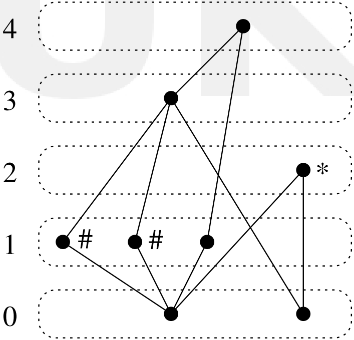

$x_8 = 2$

(2+2)-free posets $\mapsto$ ascent sequences

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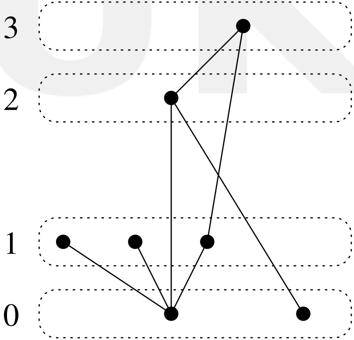

$x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, 2)$

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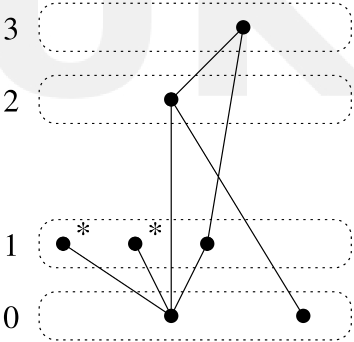

$x_7 = 1$

$x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, 2)$

(2+2)-free posets ↦ ascent sequences

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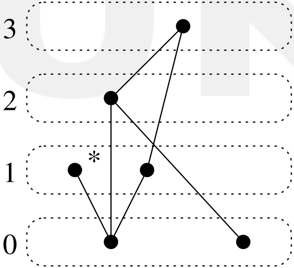

$x_6 = 1$

$x = (x_1, x_2, x_3, x_4, x_5, x_6, 1, 2)$

(2+2)-free posets $\mapsto$ ascent sequences

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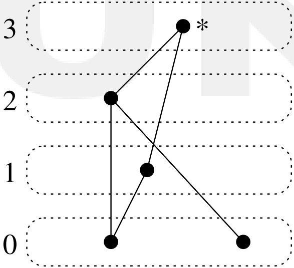
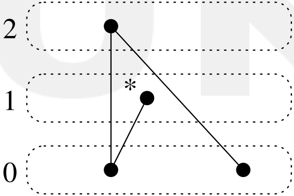


$x_5 = 3$

$x = (x_1, x_2, x_3, x_4, x_5, 1, 1, 2)$

(2+2)-free posets $\mapsto$ ascent sequences

How can one decompose such posets?
There are 3 rules ...

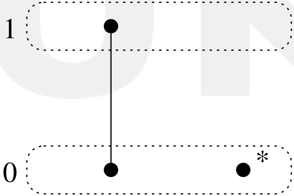


$x_4 = 1$

$x = (x_1, x_2, x_3, x_4, 3, 1, 1, 2)$

(2+2)-free posets $\mapsto$ ascent sequences

How can one decompose such posets?
There are 3 rules ...

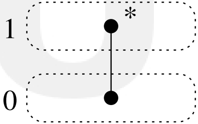


$x_3 = 0$

$x = (x_1, x_2, x_3, 1, 3, 1, 1, 2)$

(2+2)-free posets $\mapsto$ ascent sequences

How can one decompose such posets?
There are 3 rules ...

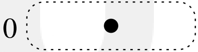


$x_2 = 1$

$x = (x_1, x_2, 0, 1, 3, 1, 1, 2)$

(2+2)-free posets $\mapsto$ ascent sequences

How can one decompose such posets?
There are 3 rules ...

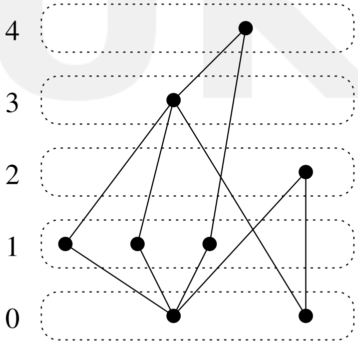


$x_1 = 0$

$x = (x_1, 1, 0, 1, 3, 1, 1, 2)$

(2+2)-free posets $\mapsto$ ascent sequences

How can one decompose such posets?
There are 3 rules ...



$\Psi(P) = (0, 1, 0, 1, 3, 1, 1, 2)$

Partition matrices

$$\begin{bmatrix} \{1,2,3\} & \emptyset & \{5,7,8\} & \{9\} \\ \emptyset & \{4\} & \{6\} & \{11\} \\ \emptyset & \emptyset & \emptyset & \{13\} \\ \emptyset & \emptyset & \emptyset & \{10,12\} \end{bmatrix}$$

Definition 2

A partition matrix on $\{1, \ldots, n\}$ is an upper triangular matrix whose elements are sets such that

(i) each column and row contain at least one non-empty set;

(ii) the non-empty sets partition $\{1, \ldots, n\}$;

(iii) $\mathrm{col}(i) < \mathrm{col}(j) \implies i < j$,

where $\mathrm{col}(i)$ denotes the column in which i is a member.

Let Par_n be the set of all such matrices.

Partition matrices

Par_3:

$$\begin{bmatrix} \{1,2,3\} \end{bmatrix} \quad \begin{bmatrix} \{1,2\} & \emptyset \\ \emptyset & \{3\} \end{bmatrix} \quad \begin{bmatrix} \{1\} & \{2\} \\ \emptyset & \{3\} \end{bmatrix}$$

$$\begin{bmatrix} \{1\} & \{3\} \\ \emptyset & \{2\} \end{bmatrix} \quad \begin{bmatrix} \{1\} & \emptyset \\ \emptyset & \{2,3\} \end{bmatrix} \quad \begin{bmatrix} \{1\} & \emptyset & \emptyset \\ \emptyset & \{2\} & \emptyset \\ \emptyset & \emptyset & \{3\} \end{bmatrix}$$

Given $A \in \mathrm{Par}_n$ and $\ell \in \{1, \ldots, n\}$, let $x_\ell = \min(A_{\star i}) - 1$ where i is the row containing ℓ and $\min(A_{\star i})$ is the smallest entry in column i of A. Define

$$\Lambda(A) = (x_1, \ldots, x_n).$$

Some properties

Theorem 3

$\Lambda : \mathrm{Par}_n \to \mathcal{I}_n$ *is a bijection*

Proposition 4

The statistic $\dim$ *on* Par_n *is Eulerian.*

Theorem 5

Let Mono_n *be the set of matrices in* Par_n *which satisfy:*

(iv) $\mathrm{row}(i) < \mathrm{row}(j) \implies i < j$.

Then

$$\Lambda(\mathrm{Mono}_n) = \text{non-decreasing inversion tables in } \mathcal{I}_n,$$
$$|Mono_n| = n\text{th Catalan number.}$$

Diagonal partition matrices

Let $\mathrm{RLE}(w)$ denote the run-length encoding of the inversion table w. For example,

$$\mathrm{RLE}(0,0,0,0,1,1,0,2,3,3) = (0,4)(1,2)(0,1)(2,1)(3,2).$$

A sequence of positive integers $(u_1, \ldots, u_k)$ which sum to n is called an integer composition of n and we write this as $(u_1, \ldots, u_k) \models n$.

Theorem 6

The set of diagonal matrices in Par_n is the image under Λ of

$$\{\, w \in \mathcal{I}_n : (u_1, \ldots, u_k) \models n \text{ and } \mathrm{RLE}(w) = (p_0, u_1) \ldots (p_{k-1}, u_k) \,\},$$

where $p_0 = 0$, $p_1 = u_1$, $p_2 = u_1 + u_2$, $p_3 = u_1 + u_2 + u_3$, etc.

Bidiagonal partition matrices

Let BiPar_n be the set of bidiagonal matrices in Par_n and

$$f(x,q) = \sum_{n\geq 0} \sum_{A\in \mathrm{BiPar}_n} q^{\dim(A)} x^n$$

Theorem 7

$$f(x,q) = \frac{2x^3 - (q+5)x^2 + (q+4)x - 1}{2(q^2+q+1)x^3 - (q^2+4q+5)x^2 + 2(q+2)x - 1}.$$

Note that

$f(x,1) = |\,\mathfrak{S}_{\mathrm{n}}(3214, 2143, 24135, 41352, 14352, 13542, 13524)|$,

the collection of permutations sortable by two pop-stacks in parallel, see Atkinson & Sack 1999.

Composition matrices

$$\begin{bmatrix} \{1,4\} & \emptyset & \{5,7,8\} & \{9\} \\ \emptyset & \{2,3\} & \{6\} & \{11\} \\ \emptyset & \emptyset & \emptyset & \{13\} \\ \emptyset & \emptyset & \emptyset & \{10,12\} \end{bmatrix}$$

Definition 8

A composition matrix on $\{1,\ldots,n\}$ is an upper triangular matrix whose elements are sets such that

(i) each column and row contain at least one non-empty set,

(ii) the non-empty sets partition $\{1,\ldots,n\}$.

Let Comp_n be the set of all such matrices.

Composition matrices

$$\mathrm{Comp}_2 \;=\; \left\{ \left[\, \{1,2\} \,\right],\;\; \begin{bmatrix} \{1\} & \emptyset \\ & \{2\} \end{bmatrix},\;\; \begin{bmatrix} \{2\} & \emptyset \\ & \{1\} \end{bmatrix} \right\}.$$

Composition matrices $\mapsto$ ascent seq $\times$ set partition

Bijection $f : \mathrm{Comp}_n \to (a, E)$

$$A = \begin{bmatrix} \{3,8\} & \{6\} & \emptyset \\ \emptyset & \{2,5,7\} & \emptyset \\ \emptyset & \emptyset & \{1,4\} \end{bmatrix}.$$

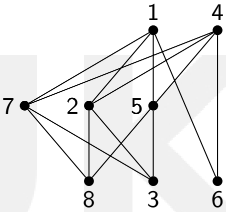
We have

$$\mathrm{Card}(A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad T_{\mathrm{Card}(A)} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and

$$\begin{aligned} f(A) &= \big(\, \Gamma(\mathrm{Card}(A)),\ E(A)\,\big) \\ &= \big(\, (0,0,1,1,1,0,2,2),\ \{3,8\}\{2,5,7\}\{6\}\{1,4\}\,\big). \end{aligned}$$

Labeled (2+2)-free posets $\mapsto$ ascent seq $\times$ set partition

$P =$ 

$\in \mathcal{P}_8$.

The unlabeled poset corresponding to P has ascent sequence $(0,0,1,1,1,0,2,2)$. There are four runs in this ascent sequence. The first run of two 0s inserts the elements 3 and 8, so we have $X_1 = \{3,8\}$. Next the run of three 1s inserts elements 2, 5 and 7, so $X_2 = \{2,5,7\}$. The next run is a run containing a single 0, and the element inserted is 6, so $X_3 = \{6\}$. The final run of two 2s inserts elements 1 and 4, so $X_4 = \{1,4\}$. Hence

$$\psi(P) = \big(\,(0,0,1,1,1,0,2,2),\ \{3,8\}\{2,5,7\}\{6\}\{1,4\}\,\big).$$

Some properties of composition matrices

- Non-decreasing ascent sequences $\times$ set partition $\Leftrightarrow$ Parking functions
- There are $k!S(n,k)$ diagonal composition matrices $A \in \mathrm{Comp}_n$ with $\dim(A) = k$.
- Let BiComp_n be the set of bidiagonal matrices in Comp_n. Then

$$\sum_{n\geq 0} \sum_{A\in \mathrm{BiComp}_n} q^{\dim(A)} \frac{x^n}{n!} = \frac{qe^{2x} - qe^x - 1}{(1-q)qe^{2x} + 2q^2e^x - q^2 - q - 1}.$$

Number of labeled (2+2)-free posets

From the original paper the g.f. for the number of (2+2)-free posets was shown to be

$$P(t) = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1-(1-t)^i\right)$$
$$= 1 + t + 2t^2 + 5t^3 + 15t^4 + 53t^5 + 217t^6 + O(t^7).$$

The exponential generating function for $(2+2)$-free posets is

$$L(t) = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1-e^{-ti}\right)$$
$$= 1 + t + 3\frac{t^2}{2!} + 19\frac{t^3}{3!} + 207\frac{t^4}{4!} + 3451\frac{t^5}{5!} + 81663\frac{t^6}{6!} + O(t^8).$$