Pairings on Bit Strings

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Pairing

A pairing on the set $\{(10)^n\} = \{1, 0, 1, 0 \cdots, 1, 0\}$ is a collection of n pairs such that each 1 must pair to a 0. We use Π_n denote the set of all pairings on $\{(10)^n\}$.

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Given a pairing $\pi \in \Pi_n$, we can represent π by a graph with $2n$ points, whose edge set consists of arcs connecting 1 and 0. For example, the figure illustrates a pairing

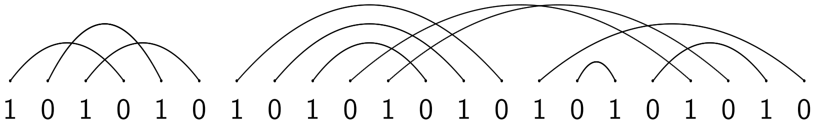

Figure: A pairing on $\{(10)^{11}\}$

Opener and closer

In a pairing graph, the opener means the left-hand end point of the arc, the closer means the right-hand endpoint of the arc.

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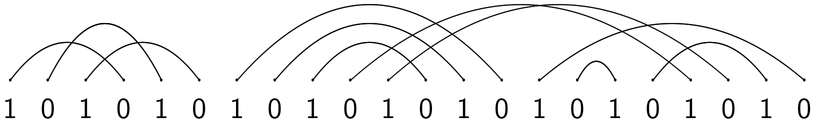

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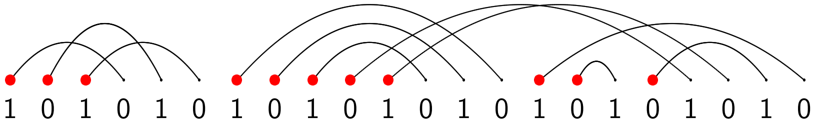

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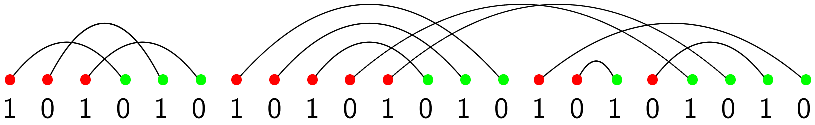
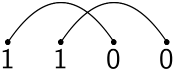
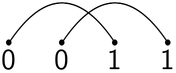


Crossing

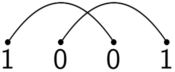
Define a crossing as a pair of crossing arcs in the graph of π. We sort the crossing into 4 types:



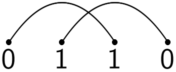
call it a crossing of type A;



call it a crossing of type B;



call it a crossing of type C;



call it a crossing of type D.

Crossing

We use $cr_A(\pi)$, $cr_B(\pi)$, $cr_C(\pi)$ and $cr_D(\pi)$ to denote the number of crossings of type A, B, C and D in π, respectively.

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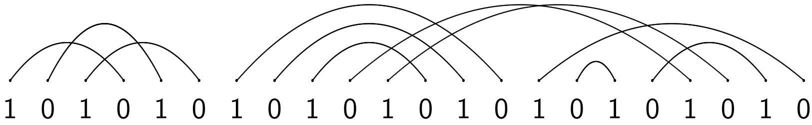

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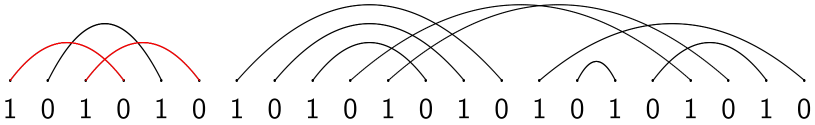

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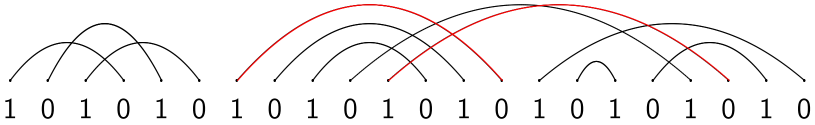

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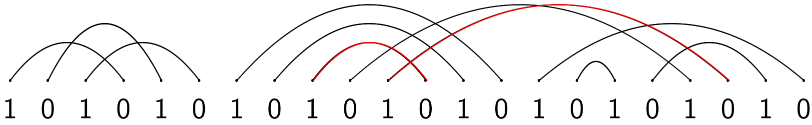

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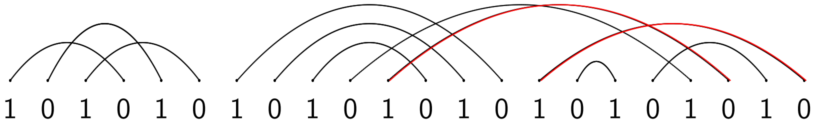

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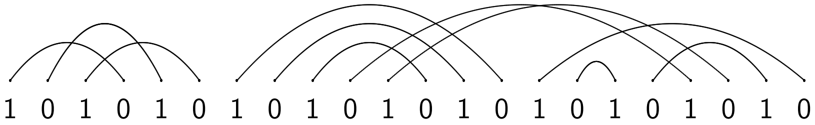

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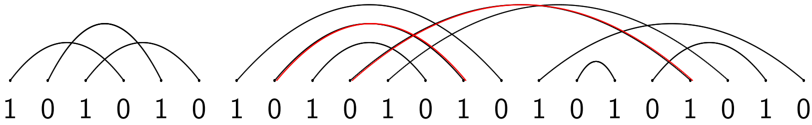

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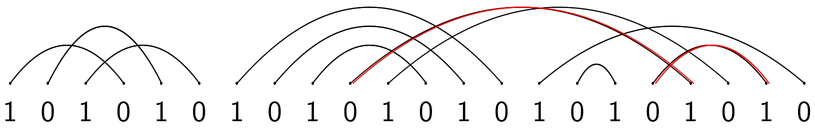

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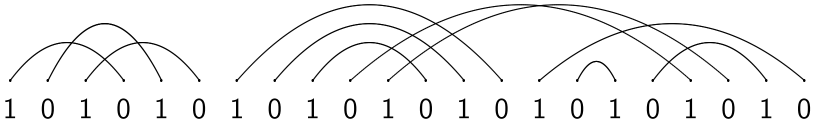

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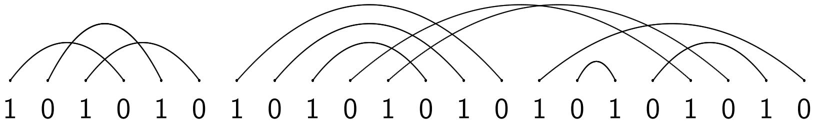
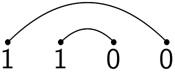


$cr_A(\pi) = 4$, $cr_B(\pi) = 2$, $cr_C(\pi) = 4$, $cr_D(\pi) = 4$

Nesting

Similarly, we define a nesting as a pair of arcs covered one by another in the graph of π. We also sort the nesting into 4 types:



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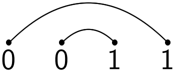
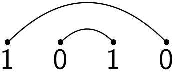
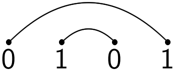


call it a nesting of type B;



call it a nesting of type C;



call it a nesting of type D.

Nesting

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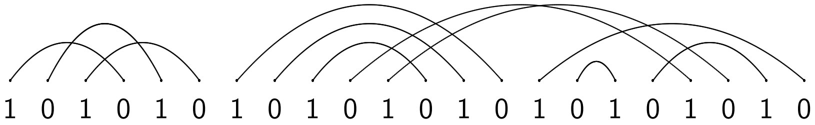

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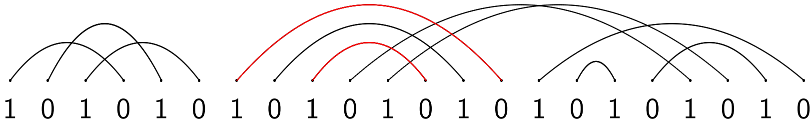

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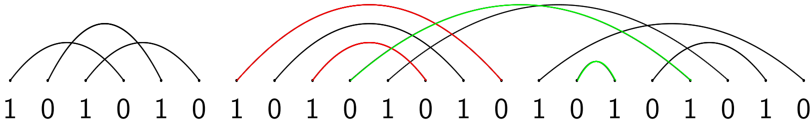

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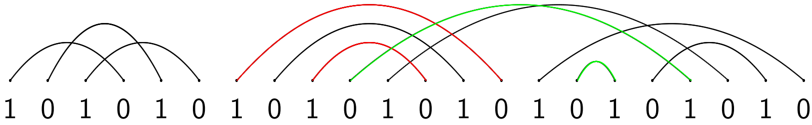

$ne_A(\pi) = 1$, $ne_B(\pi) = 1$, $ne_C(\pi) = 4$, $ne_D(\pi) = 1$.

Labeled Dyck paths

A Dyck path of semilength n is a path on the plane from the origin $(0, 0)$ to $(2n, 0)$ consisting of up steps and down steps such that the path does not go across the x-axis.

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In this paper, we will consider the Dyck path with labeling on its up steps.

We will construct a bijection ϕ between pairings on $\{(10)^n\}$ and labeled Dyck paths of semilength n, where the labeling scheme is: for an up step of hight i, it could be labeled by 0, 1, 2, $\cdots$,or $\lfloor \frac{i-1}{2} \rfloor$, called $\lfloor \frac{i-1}{2} \rfloor$ the maximal label.

Bijection ϕ

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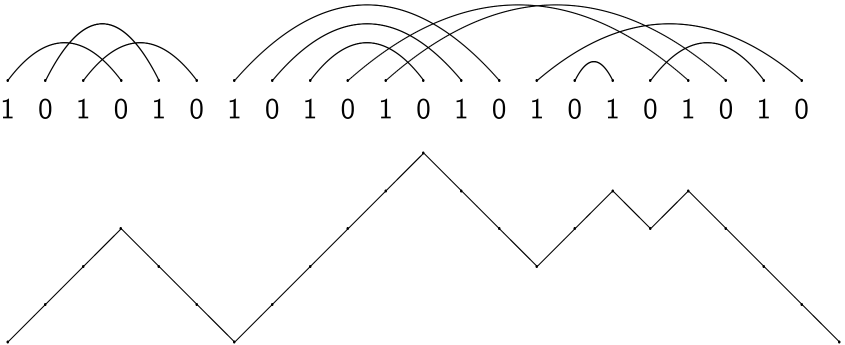
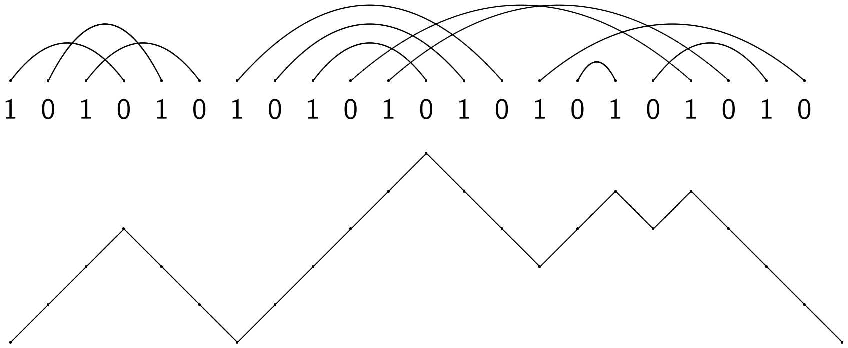


Bijection ϕ

(I) For a pairing $\pi \in \Pi_n$, each opener corresponds to an up step, and each closer corresponds to a down step.
(II) If the opener of an arc ω is 1(0), and the arc crosses with m arcs whose openers are 1(0) and located on the left of ω, then we label the corresponding up step with m.



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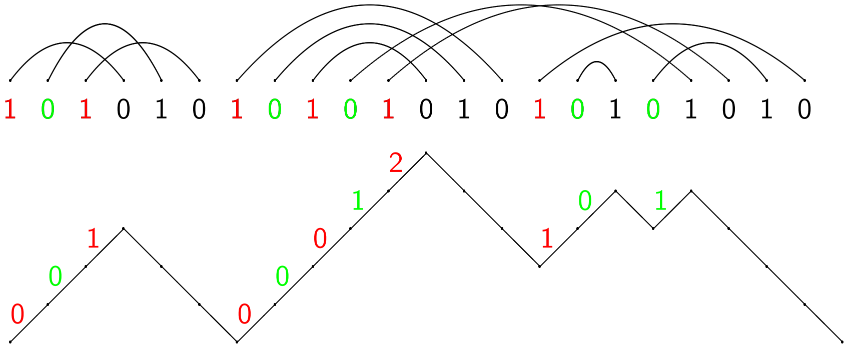

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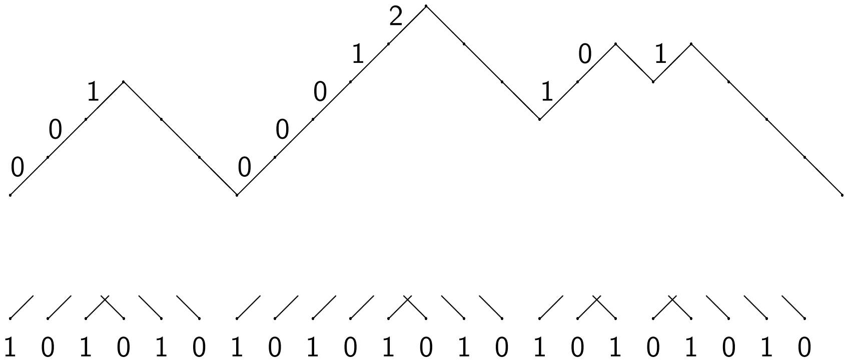

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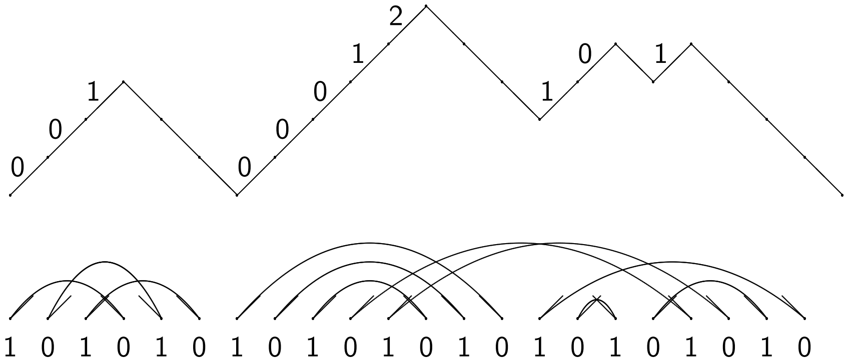

Property of the bijection ϕ

Theorem

ϕ is a bijection between pairings on $\{(10)^n\}$ and labeled Dyck paths of semilength n, where the labeling scheme is: for an up step of hight i, it could be labeled by 0, 1, 2, $\cdots$, or $\lfloor \frac{i-1}{2} \rfloor$.
Furthermore, for any pairings π, we have

$$cr_A(\pi) + ne_A(\pi) = \sum \textit{maximal label}$$

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where the sum over all up steps of odd level on $\phi(\pi)$.

$$cr_B(\pi) + ne_B(\pi) = \sum \textit{maximal label}$$

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where the sum over all up steps of even level on $\phi(\pi)$.

Property of the bijection ϕ

Corollary

The bijection ϕ on pairings preserves openers and closers and interchange the crossings and nestings of type A and B.

We derive immediately the following equality.

$$\sum x^{cr_A(\pi)} y^{cr_B(\pi)} p^{ne_A(\pi)} q^{ne_B(\pi)} = \sum x^{ne_A(\pi)} y^{ne_B(\pi)} p^{cr_A(\pi)} q^{cr_B(\pi)}$$

where the sums over all pairings with the openers sets $(\mathcal{O}_1, \mathcal{O}_0)$ and closers sets $(\mathcal{C}_1, \mathcal{C}_0)$.

Q: How about the crossing and nesting of type C and D?

Pairings and permutations

If we write down the position of each 0 which is connected to 1 in order, then we can obtain a permutation. So the total number of pairings on the set $\{(10)^n\}$ is $n!$.

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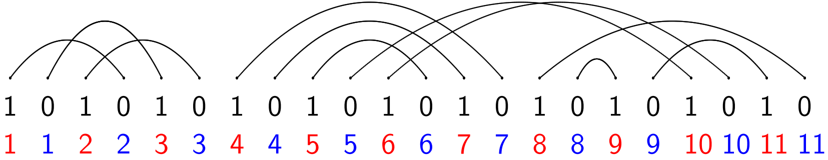

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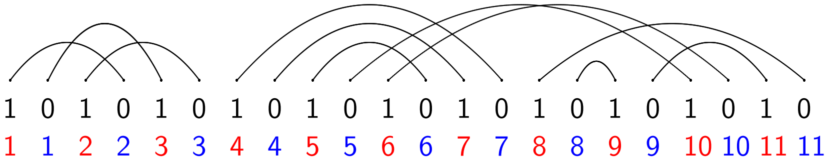

$$\pi = 2\ 3\ 1\ 7\ 6\ 10\ 4\ 11\ 8\ 5\ 9$$

Q: Which statistics do the crossing and nesting of type A, B, C and D correspond to?

Pairings without crossing of type A and set partition

The pairings without crossing of type A is corresponding to the set partition on $[n]$, and the crossing of type B on pairing is corresponding to the crossing on set partition.

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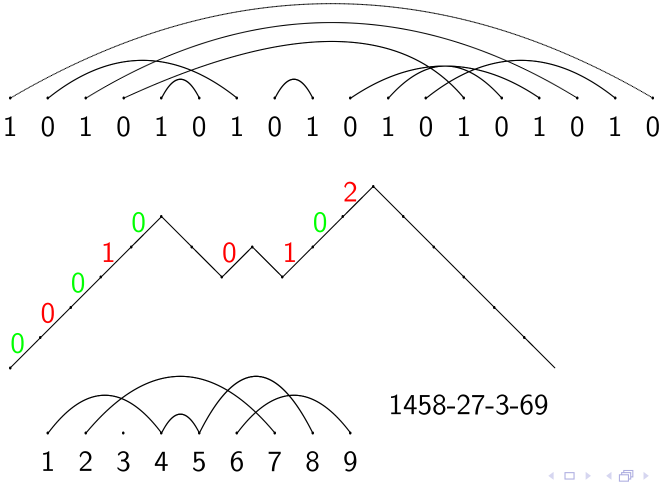

Further work

1. For the pairing, how about k-crossing and k-nesting of type A, B, C and D.

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1. For the pairing, how about k-crossing and k-nesting of type A, B, C and D.
2. For the set $\{(1100)^n\}$, and each 1 must pair to a 0, how about the property for the pairings?

Note that, the number of non-crossing pairings on the set $\{(1100)^n\}$ is

$$\frac{1}{2n+1}\binom{3n}{n}$$

Thank You