Enumerating permutations containing few copies of 321 and 3412

Dan Daly
Southeast Missouri State University

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Dartmouth College

Outline

1. Motivation
2. Connections between permutation patterns and reduced decompositions.
3. New Results

Definition

A permutation is *Boolean* if it avoids 321 and 3412.

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Theorem (Fan 1996, West 1998)

The number of permutations that avoid 321 *and* 3412 *is* F_{2n-1} *where* F_k *is the* k^{th} *Fibonacci number.*

Theorem (Egge 2003)

The number of involutions in S_n which avoid 3412 and contain exactly one copy of 321 is

$$\frac{2(n-1)F_n - nF_{n-1}}{5}$$

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To do more, we will use the relationship between permutation patterns and reduced decompositions.

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Let $\pi \in S_n$. If $\pi = s_{i_1} s_{i_2} \dots s_{i_k}$ is an expression for π of minimal length, then $(i_1, i_2, \dots, i_k)$ is a *reduced decomposition* for π.

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$s_1 s_2 s_1 = (12)(23)(12) = 321$.
$(1, 2, 1)$ is a reduced decomposition for 321.

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Braid Moves:
$(i, j) = (j, i)$ when $|i - j| > 1$.
$(i, i+1, i) = (i+1, i, i+1)$ $\forall i$.

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π avoids 321 and 3412 if and only if there exists a reduced decomposition of π with no repeated elements.

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Example

$(2, 3, 4, 1) = (23)(34)(45)(12) = 31452$ avoids 321 and 3412.

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Example

$(2,3,4,1) = (23)(34)(45)(12) = 31452$ avoids 321 and 3412.
$(2,3,4,2) = (23)(34)(45)(23) = 14352$ contains 321 and avoids 3412.

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Definition

A consecutive substring of a reduced decomposition is called a *factor* of the reduced decomposition.

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Theorem

$\pi \in S_n$ has a reduced decomposition with exactly one element repeated if and only if π avoids 3412 and contains exactly one 321 pattern or π avoids 321 and contains exactly one 3412 pattern.

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- *π contains exactly one 321 pattern and avoids 3412 if and only if π has a reduced decomposition with $(i, i+1, i)$ as a factor for some $i \in \{1, \ldots n-2\}$ and no other repetitions.*

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- *π contains exactly one 321 pattern and avoids 3412 if and only if π has a reduced decomposition with $(i, i+1, i)$ as a factor for some $i \in \{1, \dots n-2\}$ and no other repetitions.*
- *π contains exactly one 3412 pattern and avoids 321 if and only if π has a reduced decomposition with $(i, i-1, i+1, i]$ as a factor for some $i \in \{2, \dots n-2\}$ and no other repetitions.*

Examples

- 25314 has r.d. $(4, 1, 2, 3, 2)$
  ($(2, 3, 2)$ is of the form $(i, i+1, i)$, so there must be a 321.)

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  ($[2132]$ is of the form $[i(i-1)(i+1)i]$, so there must be a 3412.)

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  ($[2132]$ is of the form $[i(i-1)(i+1)i]$, so there must be a 3412.)

Theorem

The number of permutations in $Av_n(3412)$ *that contain exactly one* 321 *is equal to the number of permutations in* $Av_{n+1}(321)$ *that contain exactly one* 3412.

Theorem

The number of permutations in S_n that avoid 3412 and contain exactly one 321 is

$$\sum_{i=1}^{n-2} F_{2i}F_{2(n-i-1)}$$

where F_m is the m^{th} Fibonacci number.

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Closed Form:

$$\sum_{i=1}^{n-2} F_{2i}F_{2(n-i-1)} = \frac{2(2n-5)F_{2n-6} + (7n-16)F_{2n-5}}{5}$$

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Generating Function:

$$\frac{x^3}{(1-3x+x^2)^2}$$

n	3	4	5	6	7	8	9
	1	6	25	90	300	954	2939

(OEIS A001871)

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Theorem

If **s** *is a reduced decomposition with exactly one element occurring three times and no other repetitions, then there exists a reduced decomposition* **t** *equivalent to* **s** *with precisely two elements each repeated once and no other repetitions.*

Consider reduced decompositions with two elements each repeated once and no other repetitions.

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If a permutation has a reduced decomposition with exactly one element repeated and no other repetitions, we can use braid moves to minimize the length of the factor in between the repeated elements to either $(i, i+1, i)$ or $(i, i+1, i-1, i)$ for some i.

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What are the possible "minimal" factors for permutations with two repetitions?

Definition

If $\mathbf{s} = (i_1, \ldots, i_m)$ is a reduced decomposition of $\pi \in S_n$ then a *repetition factor* of $\mathbf{s}$ is a factor $(i_j, \ldots, i_k)$ of $\mathbf{s}$ with $1 \leq j < k \leq m$ such that all elements that occur more than once in $\mathbf{s}$ occur in the factor $(i_j, \ldots i_k)$.

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Definition

Let $\mathbf{s} = (i_1 \ldots i_m)$ be a reduced decomposition. A repetition factor $(i_j, \ldots i_k)$ is *minimal* if $v - u$ is minimal among all equivalent reduced decompositions for $\mathbf{s}$.

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Example

$(1, 2, 3, 4, 2)$ has repetition factor $(2, 3, 4, 2)$, but since $(1, 2, 3, 4, 2)$ is equivalent to $(1, 2, 3, 2, 4)$ the repetition factor is not minimal.

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Example

$(1, 2, 3, 4, 2)$ has repetition factor $(2, 3, 4, 2)$, but since $(1, 2, 3, 4, 2)$ is equivalent to $(1, 2, 3, 2, 4)$ the repetition factor is not minimal. $(2, 3, 2)$ is a minimal repetition factor.

Definition

A minimal repetition factor $(i_1, \ldots i_k)$ with two repetitions is *entangled* if it is not equivalent to a factor of the form $(p, \ldots, p, \ldots, q, \ldots, q)$. A minimal repetition factor with two repetitions is *unentangled* if it is equivalent to a factor of the form $(p, \ldots, p, \ldots, q, \ldots, q)$.

Classification of Entangled Factors

Entangled Factors of Length 5

- $(i, i-1, i+1, i, i+1)$ $(2,1,3,2,3) \rightarrow 3421$
- $(i+1, i, i+1, i-1, i)$ $(3,2,3,1,2) \rightarrow 4312$
- $(i+1, i, i-1, i, i+1)$ $(3,2,1,2,3) \rightarrow 4231$

Classification of Entangled Factors

Entangled Factors of Length > 5

Length 6

- $(i, i-1, i+1, i, i+2, i+1)$ $(2, 1, 3, 2, 4, 3) \to 34512$
- $(i+1, i+2, i, i+1, i-1, i)$ $(3, 4, 2, 3, 1, 2) \to 45123$
- $(i, i-1, i+1, i+2, i+1, i)$ $(2, 1, 3, 4, 3, 2) \to 35142$
- $(i+1, i+2, i, i-1, i, i+1)$ $(3, 4, 2, 1, 2, 3) \to 42513$

Length 7

- $(i, i-1, i+2, i+1, i+3, i+2, i)$ $(2, 1, 4, 3, 5, 4, 2) \to 351624$

Theorem

A factor of a reduced decomposition with exactly two repeated elements has a subfactor that is equivalent to one of the previous entangled factors or has a subfactor that is of the form $(p, \ldots, p, \ldots, q, \ldots, q)$. *In particular, any entangled factor is equivalent to one that has been previously listed.*

Want: Connection between reduced decompositions with two elements each repeated once and no other repetitions and pattern conditions involving 321 and 3412.

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Focus on the case of permutations containing exactly 2 321 patterns and avoiding 3412. There are similar results for containing exactly one 321 and exactly one 3412 pattern and for containing exactly two 3412 patterns and avoiding 321.

A permutation in $Av_n(3412)$ can contain exactly two 321 patterns by sharing:

- 2 elements: $\{3421, 4312, 4231\}$

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- 2 elements: $\{3421, 4312, 4231\}$
- 1 element: $\{32541, 52143\}$
- 0 elements: $\{321654, 326154, 421653\}$

Theorem

π has a reduced decomposition with a factor of the form

1. $(i, i-1, i+1, i, i+1)$
2. $(i+1, i, i+1, i-1, i)$
3. $(i+1, i, i-1, i, i+1)$

and no other repetitions if and only if π has exactly two 321 patterns of the corresponding form

1. 3421
2. 4312
3. 4231

and avoids 3412.

Theorem

π has a reduced decomposition with a factor of the form

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2. $(i+1, i, i+1, i-1, i)$
3. $(i+1, i, i-1, i, i+1)$

and no other repetitions if and only if π has exactly two 321 *patterns of the corresponding form*

1. 3421
2. 4312
3. 4231

and avoids 3412.

Note: this takes care of the entangled factors for this case. What about the unentangled factors?

Theorem

$\pi \in S_n$ has a reduced decomposition with a factor of the form

1. $(i, i+1, i, i+2, i+3, \ldots, i+k, i+k+1, i+k)$
2. $(i, i+1, i, i-1, i-2, \ldots, i-k, i-k-1, i-k)$

and no other repetitions if and only if π has exactly two 321 *patterns of the form*

1. 32541
2. 52143

and avoids 3412.

Theorem

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and no other repetitions if and only if π has exactly two 321 patterns of the form

1. 32541
2. 52143

and avoids 3412.

Example

$(2, 3, 2, 4, 5, 6, 7, 6)$ gives the permutation 14356872

Theorem

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and no other repetitions if and only if π has exactly two 321 patterns of the form

1. 32541
2. 52143

and avoids 3412.

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$(2, 3, 2, 4, 5, 6, 7, 6)$ gives the permutation 14356872

Theorem

$\pi \in S_n$ has a reduced decomposition with a factor of the form $(i, i+1, i, j, j+1, j)$ where $|i-j| > 2$ and no other repetitions if and only if π has exactly two 321 patterns of the form 321654, 326154 or 421653 and avoids 3412.

Theorem

The following quantities are equal:

- $|\{\pi \in S_n :$ π *avoids 3412 and contains exactly two 321 patterns of the form 3421*$\}|$
- $|\{\pi \in S_n :$ π *avoids 3412 and contains exactly two 321 patterns of the form 4312*$\}|$
- $|\{\pi \in S_n :$ π *avoids 3412 and contains exactly two 321 patterns of the form 4231*$\}|$
- $\sum_{i=1}^{n-3} F_{2i} F_{2(n-i-2)}$ *where* F_m *is the* m^{th} *Fibonacci number*

Theorem

The following quantities are equal:

- $|\{\pi \in Av_n(3412) :$ π *contains exactly two 321 patterns of the form 32541*$\}|$
- $|\{\pi \in Av_n(3412) :$ π *contains exactly two 321 patterns of the form 52143*$\}|$
- $$\sum_{k=3}^{n-2}\left(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)}\right)$$

where F_m is the m^{th} Fibonacci number

Let

$$f(a) = F_{2a+1} + 2\sum_{m=1}^{a} F_{2(a-m)+1} + (a-2) + \sum_{m=1}^{a-2}(a-m-1)F_{2m+1}$$

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Theorem

The number of permutations in $Av_n(3412)$ containing exactly two 321 patterns sharing no elements is

$$\sum_{k=4}^{n-2} f(k-3)\left(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)}\right)$$

Theorem

The number of permutations in $Av_n(3412)$ *that contain exactly two* 321 *patterns is*

$$3\sum_{k=1}^{n-3} F_{2k}F_{2(n-k-2)} + 2\sum_{k=3}^{n-2}\Big(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)}\Big)+$$

$$\sum_{k=4}^{n-2} f(k-3)\Big(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)}\Big)$$

n	4	5	6	7	8	9	10	11	12
	3	20	92	363	1317	4530	15012	48391	152674

Theorem

The number of permutations in S_n that contain exactly one 321 *pattern and exactly one* 3412 *pattern is*

$$2\sum_{k=1}^{n-4} F_{2k}F_{2(n-k-3)} + 4\sum_{k=4}^{n-2}\left(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)}\right)+$$

$$2\sum_{k=5}^{n-2} f(k-4)\left(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)}\right)$$

n	5	6	7	8	9	10	11	12
	2	16	84	366	1434	5244	18268	61382

Theorem

The number of permutations in $Av_n(321)$ *that contain exactly two* 3412 *patterns is*

$$\sum_{k=1}^{n-5} F_{2k}F_{2(n-k-4)} + 2\sum_{k=5}^{n-2}\Big(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)}\Big)+$$

$$\sum_{k=6}^{n-2} f(k-5)\Big(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)}\Big)$$

n	6	7	8	9	10	11	12
	1	8	42	183	717	2622	9134

Thank you!