Expanding permutation statistics as sums of permutation patterns

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Permutations/patterns as functions

Think of $\pi \in \mathfrak{S}$ as a function $\pi : \mathfrak{S} \to \mathbb{N}$ that counts occurrences of π

Example

- $1 = | \cdot |$
- $21 = \mathrm{inv}$

Statistics as linear combinations of patterns

Any function

$$\mathsf{stat} : \mathfrak{S} \to \mathbb{C}$$

may be represented uniquely as a (typically infinite) sum

$$\mathsf{stat} = \sum_{\pi \in \mathfrak{S}} \lambda(\pi)\pi$$

where $\{\lambda(\pi)\}_{\pi \in \mathfrak{S}} \subset \mathbb{C}$

Example

$$\mathrm{lmax} \;=\; \sum_{\substack{\pi \in \mathfrak{S} \\ \pi(|\pi|)=1}} (-1)^{|\pi|-1}\pi \;=\; 1 - 21 + 231 + 321 - \cdots$$

The incidence algebra $I(P)$

Let Q be a locally finite poset.

The incidence algebra, $I(Q)$, is the $\mathbb{C}$-algebra of all functions $Q \times Q \to \mathbb{C}$ with multiplication

$$(FG)(x,z) = \sum_{x \le y \le z} F(x,y)G(y,z)$$

and identity

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y \end{cases}$$

The incidence algebra $I(\mathfrak{S})$

- Define $\pi \leq \sigma$ in $\mathfrak{S}$ if $\pi(\sigma) > 0$
- Define $P \in I(\mathfrak{S})$ by $P(\pi, \sigma) = \pi(\sigma)$

$$
\begin{array}{c|ccccccccccc}
 & \epsilon & 1 & 12 & 21 & 123 & 132 & 213 & 231 & 312 & 321 & \cdots \\
\hline
\epsilon & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & \cdots \\
12 & 0 & 0 & 1 & 0 & 3 & 2 & 2 & 1 & 1 & 0 & \cdots \\
21 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & \cdots \\
123 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
132 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
213 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
231 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
312 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
321 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

The incidence algebra $I(\mathfrak{S})$

- Define $\pi \leq \sigma$ in $\mathfrak{S}$ if $\pi(\sigma) > 0$
- Define $P \in I(\mathfrak{S})$ by $P(\pi, \sigma) = \pi(\sigma)$

P is invertible because $P(\pi, \pi) = 1$.

Therefore, for any stat : $\mathfrak{S} \to \mathbb{C}$, there are unique scalars $\{\lambda(\sigma)\}_{\sigma \in \mathfrak{S}} \subset \mathbb{C}$ such that

$$\mathrm{stat} = \sum_{\sigma \in \mathfrak{S}} \lambda(\sigma)\sigma. \tag{1}$$

Indeed, $I(\mathfrak{S})$ acts on the right of $\mathbb{C}^{\mathfrak{S}}$ by

$$(f * F)(\pi) = \sum_{\sigma \leq \pi} f(\sigma)F(\sigma, \pi).$$

Thus (1) iff $\mathrm{stat} = \lambda * P$ iff $\lambda = \mathrm{stat} * P^{-1}$.

$$\text{lmax} = 1 - 21 + 231 + 321 - \cdots$$

Why?

$$\mathrm{lmax} = 1 - 21 + 231 + 321 - \cdots$$

Why?

des, maj, exc, fix, ...

How do we expand them?

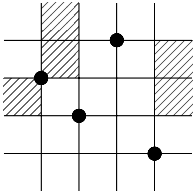
Mesh patterns

A mesh pattern is a pair

$$p = (\pi, R) \;\text{ with } \pi \in \mathfrak{S}_k \text{ and } R \subseteq [0,k] \times [0,k]$$

Example

$$p = \big(\, 3241,\ \{(0,2),(1,3),(1,4),(4,2),(4,3)\}\,\big)$$

$=$ 

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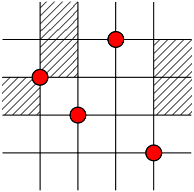
$=$ 

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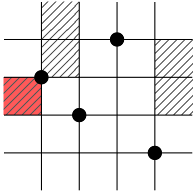
$=$ 

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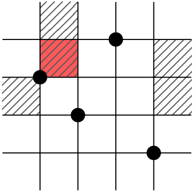
$=$ 

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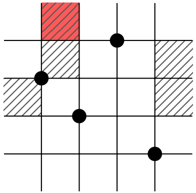
$$p = (\,3241,\ \{(0,2),(1,3),(1,4),(4,2),(4,3)\}\,)$$

$=$ 

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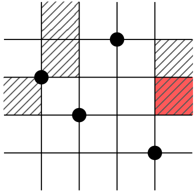
$=$ 

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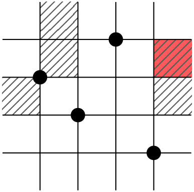
$=$ 

What is an occurrence of a mesh pattern?

$p = (\pi, R) : \mathfrak{S} \to \mathbb{N}$

$p(\tau)$ is

- the number of "classical" occurrences of π in τ such that
- no elements of τ are in the shaded regions defined by R

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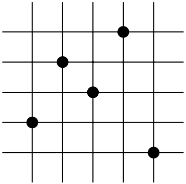
$p(\,2\,4\,3\,5\,1\,) \;=\; p\Bigg($  $\Bigg) = 3$

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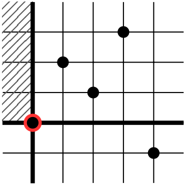
$p($2 4 3 5 1$) = p$  $= 3$

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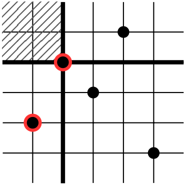
$p(24351) = p\Bigg($  $\Bigg) = 3$

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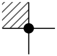
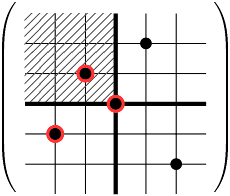
With $p = (1, \{(0,1)\}) =$  we have

$p(2\,4\,3\,5\,1) = p$  $= 3$

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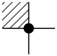
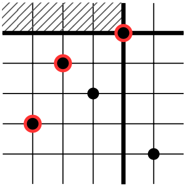
With $p = (1, \{(0,1)\}) =$  we have

$p($ 2 4 3 5 1 $) = p\Bigg($  $\Bigg) = 3$

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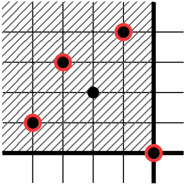
$p(24351) = p\Big($  $\Big) = 3$

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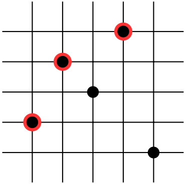
$p(\,2\,4\,3\,5\,1\,) \;=\; p\Bigg($  $\Bigg) = 3$

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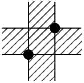
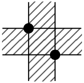
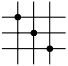
Example (Chayne Homsberger)

\# consecutive adjacent entries in τ =

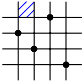
( + ) (τ)

$p = (\pi, R) \in \mathfrak{S}_k \times [0,k]^2$

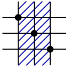
classic: $R = \emptyset$



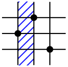
barred: $R = \{(i-1, \pi(i)-1)\}$ $\quad 3\bar{5}241 =$



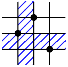
segment: $R = [1, k-1]^2$ $\quad \text{ddes} =$



dashed/vincular: $R = \cup$ vertical strips $\quad$ 23-1 $=$



bivincular: $R = \cup$ vertical and horizontal strips



The reciprocity theorem

Theorem (Reciprocity)

For any mesh pattern $p = (\pi, R)$ we have

$$p = \sum_{\sigma \in \mathfrak{S}} (-1)^{|\sigma| - |\pi|} p^\star(\sigma)\sigma$$

Definition (Dual pattern)

For $p = (\pi, R)$ let $p^\star = (\pi, R^c)$, where $R^c = [0, |\pi|]^2 \setminus R$:

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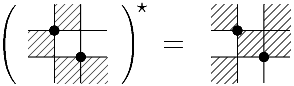
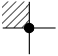
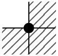


We can now explain why

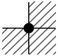
$$\mathrm{lmax} = 1 - 21 + 231 + 321 - \cdots$$

We have $\mathrm{lmax} =$  and $\Big($  $\Big)^{\star} =$ .

Thus

$$\mathrm{lmax} = \sum_{\sigma \in \mathfrak{S}} \lambda(\sigma)\sigma$$

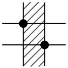
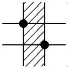
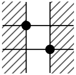
where

$\lambda(\sigma) = (-1)^{|\sigma|-1}$  (σ)

$$= \begin{cases} (-1)^{|\sigma|-1} & \text{if } \sigma(|\sigma|) = 1, \\ 0 & \text{otherwise} \end{cases}$$

Also,

$$\mathrm{des} = 21 - 231 - 312 - 321 + \cdots$$

We have des =  and $\Big($  $\Big)^{\star}$ =  .

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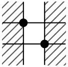
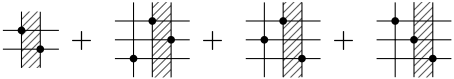
$\lambda(\sigma) = (-1)^{|\sigma|}$  (σ)

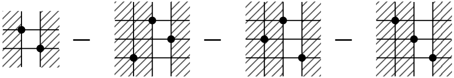
$$= \begin{cases} (-1)^{|\sigma|} & \text{if } \sigma(1) > \pi(|\sigma|), \\ 0 & \text{otherwise} \end{cases}$$

Babson and Steingrímsson classified Mahonian statistics using patterns. For instance

$$\mathrm{maj} = (21) + (1\text{-}32) + (2\text{-}31) + (3\text{-}21)$$

$=$ 

Thus we may write $\mathrm{maj} = \sum_{\pi \in \mathfrak{S}} \lambda(\pi)\pi$ where

$(-1)^{|\cdot|}\lambda(\,\cdot\,) =$ 

This last expression simplifies to

$$\lambda(\pi) = \begin{cases} 1 & \text{if } \pi = 21 \\ (-1)^n & \text{if } \pi(2) < \pi(n) < \pi(1) \\ (-1)^{n+1} & \text{if } \pi(1) < \pi(n) < \pi(2) \\ 0 & \text{otherwise} \end{cases}$$

where $n = |\pi|$

If $a_1 \ldots a_n \in \mathfrak{S}_n$, then a_i is called a strong fixed point if

- $j < i \implies a_j < a_i$ and
- $j > i \implies a_j > a_i$.

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Let

- $\mathrm{sfix}(\tau) = \#$ strong fixed points of τ
- $\mathrm{ssfix}(\tau) = \#$ strong fixed points of τ^r (skew sfix)

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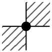
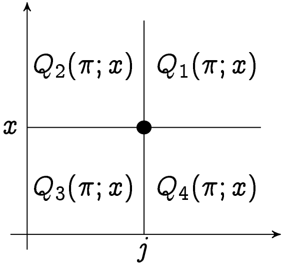
Theorem

$$\mathrm{sfix} = \sum_{\pi} (-1)^{|\pi|-1}\, \mathrm{ssfix}(\pi)\pi$$

Proof.

sfix =  and ssfix = 



$$Q_1(\pi;x) = \{\, \pi(i) : i > j,\ \pi(i) > x \,\}$$
$$Q_2(\pi;x) = \{\, \pi(i) : i < j,\ \pi(i) > x \,\}$$
$$Q_3(\pi;x) = \{\, \pi(i) : i < j,\ \pi(i) < x \,\}$$
$$Q_4(\pi;x) = \{\, \pi(i) : i > j,\ \pi(i) < x \,\}$$

The point x in π is

- a fixed point if $|Q_2(\pi;x)| = |Q_4(\pi;x)|$
- an excedance if $|Q_4(\pi;x)| > |Q_2(\pi;x)|$

A corollary to the reciprocity theorem

Recall that $P(\pi, \tau) = \pi(\tau)$.

Theorem (Inverse)

The inverse of P in $I(\mathfrak{S})$ is given by

$$P^{-1}(\pi, \tau) = (-1)^{|\tau| - |\pi|} P(\pi, \tau)$$

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Proof.

For $\pi \in \mathfrak{S}_k$, let $p = (\pi, [0,k] \times [0,k])$. Then $p^\star = (\pi, \emptyset)$ and

$$p(\tau) = \sum_{\sigma \in \mathfrak{S}} (-1)^{|\sigma| - |\pi|} p^\star(\sigma)\sigma(\tau) \qquad \text{(reciprocity)}$$

Thus

$$\delta(\pi, \tau) = \sum_{\pi \leq \sigma \leq \tau} (-1)^{|\sigma| - |\pi|} P(\pi, \sigma) P(\sigma, \tau)$$

By the inverse theorem (but not trivially):

$$\mathrm{fix} = \sum_{\pi} \left((-1)^{|\pi|-1} \sum_{x \in \mathrm{SSF}(\pi)} \binom{|\pi|-1}{x-1} \right) \pi$$

$$\mathrm{exc} = \sum_{\pi} \left((-1)^{|\pi|-2} \sum_{x \in \mathrm{SSF}(\pi)} \binom{|\pi|-2}{x-2} \right) \pi$$

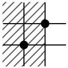
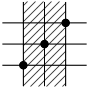
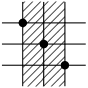
where $\mathrm{SSF}(\pi)$ is the set of skew strong fixed points in π

Alternating permutations and the Euler numbers

A permutation $\pi \in \mathfrak{S}_n$ is said to be alternating if

$$\pi(1) > \pi(2) < \pi(3) > \pi(4) < \cdots$$

Alternating permutations are those that avoid

,  and 

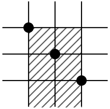
In 1879, André showed that the number of alternating permutations in $\mathfrak{S}_n$ is the Euler number E_n given by

$$\sum_{n \geq 0} E_n x^n / n! = \sec x + \tan x.$$

Simsun permutations

A permutation $\pi \in \mathfrak{S}_n$ is simsun if for all $i \in [1, n]$, after removing the i largest letters of π, the remaining word has no double descents.

A permutation is simsun if and only if it avoids the pattern

$\mathfrak{simsun} =$ 

The number of simsun permutations in $\mathfrak{S}_n$ is E_{n+1}.

André permutations

André permutations of various kinds were introduced by Foata and Schützenberger and further studied by Foata and Strehl.

If $\pi \in \mathfrak{S}_n$ and $x = \pi(i) \in [1, n]$ let $\lambda(x), \rho(x) \subseteq [1, n]$ be defined as follows. Let $\pi(0) = \pi(n+1) = -\infty$.

- $\lambda(x) = \{\pi(k) : j_0 < k < i\}$ where
  $j_0 = \max\{j : j < i \text{ and } \pi(j) < \pi(i)\}$, and
- $\rho(x) = \{\pi(k) : i < k < j_1\}$ where
  $j_1 = \min\{j : i < j \text{ and } \pi(j) < \pi(i)\}$.

Then π is an André permutation of the first kind if

$$\max \lambda(x) \leq \max \rho(x)$$

for all $x \in [1, n]$, where $\max \emptyset = -\infty$.

In particular, π has no double descents and $\pi(n) = n$.

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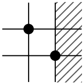
André permutations

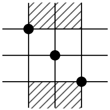
Fact: There are exactly E_n André permutations in $\mathfrak{S}_n$.

Theorem

Let $\pi \in \mathfrak{S}_n$. Then π is an André permutation of the first kind if and only if it avoids



and



$= \mathfrak{andr\acute{e}}$

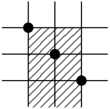
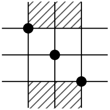
Corollary

$$\mathfrak{S}_n(\mathfrak{andr\acute{e}}) = E_{n+1}$$

A Wilf-equivalence

Corollary

The two patterns

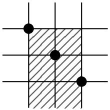
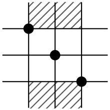
$\mathfrak{simsun} =$  *and* $\mathfrak{andr\acute{e}} =$ 

are Wilf-equivalent

A Wilf-equivalence

Corollary

The two patterns

$\mathfrak{simsun} =$  *and* $\mathfrak{andr\acute{e}} =$ 

are Wilf-equivalent

Moreover, these are the only essentially different patterns in this Wilf-class

We have barely scratched the surface.

For $n = 3$ we have

$$\begin{aligned} \#\ \text{patterns} &= 393216 \\ \#\ 2\ \text{element sets of patterns} &= 77309214720 \\ \#\ 3\ \text{element sets of patterns} &= 10133021852303360 \\ \#\ 4\ \text{element sets of patterns} &= 996108980402440273920 \\ &\vdots \end{aligned}$$

$p = (\pi, R) \in \mathfrak{S}_k \times [0, k]^2$

Restrict R?

- R as a relation: reflexive, symmetric, transitive, ...
- R as a digraph: acyclic, rooted tree, tournament, ...