The weak Bruhat order and separable permutations

Fan Wei Massachusetts Institute of Technology

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $1, 2, \ldots n$, partially ordered by the weak (Bruhat) order. Thus for any permutation $w \in \mathfrak{S}_n$, the rank $\ell(w)$ of w in $\mathfrak{S}_n$ is the number of inversions in w. It follows that the rank-generating function of $\mathfrak{S}_n$ is

$$F(\mathfrak{S}_n, q) = \sum_{w \in \mathfrak{S}_n} q^{\ell(w)} = (\boldsymbol{n})!,$$

where $(\boldsymbol{n})! = (\boldsymbol{1})(\boldsymbol{2}) \cdots (\boldsymbol{n})$ and $(\boldsymbol{i}) = 1 + q + q^2 + \cdots + q^{i-1}$.

For any $w \in \mathfrak{S}_n$, we define two graded posets associated with w: $\Lambda_w = \{v \in \mathfrak{S}_n : v \leq w\}$ and $V_w = \{v \in \mathfrak{S}_n : v \geq w\}$. In V_w, we define the rank of $v \in V_w$ to be $\ell(v) - \ell(w)$.

In general the rank-generating functions of Λ_w and V_w are messy. However, in this talk we will show that if w is a separable permutation, i.e., 3142-avoiding and 2413-avoiding, then there is the surprising formula

$$F(\Lambda_w, q) F(V_w, q) = (\boldsymbol{n})!.$$

Moreover, we define a bijection $\varphi \colon \Lambda_w \times V_w \to \mathfrak{S}_n$ satisfying $\ell(u) + \ell(v) - \ell(w) = \ell(\varphi(u, v))$, and we give a convenient method to find an explicit formula for $F(\Lambda_w, q)$ and $F(V_w, q)$. In particular, we find that if w is a 231-avoiding, or 213-avoiding, or 312-avoiding, or 132-avoiding permutation, a more direct expression for $F(\Lambda_w, q)$ and $F(V_w, q)$ can be given. We also deduce from our formula for $F(\Lambda_w, q)$ and $F(V_w, q)$ that the posets Λ_w and V_w are rank-symmetric and rank-unimodal for all separable permutation w.

These results were obtained under the supervision of Richard Stanley when the author was an undergraduate at M.I.T.