# On the size of sets of permutations with bounded VC-dimension

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A family $\mathcal{P}$ of $n$-permutations has VC-dimension $k$ if $k$ is the largest number such that the elements of $\mathcal{P}$ induce all $k!$ permutations on some $k$-tuple of indices. An example of a family with VC-dimension $k$ are permutations avoiding some fixed $(k+1)$-permutation. Marcus and Tardos proved the Stanley-Wilf conjecture, which says that the number of $n$-permutations avoiding an arbitrary given permutation grows only exponentially in $n$. Raz showed that if a family of $n$-permutations has VC-dimension 2, then its size is at most exponential in $n$.

We show that every set of permutations with VC-dimension $k$ has size at most $2^{O(n \log^\star(n))}$, where the constant in the O-notation depends only on $k$. On the other hand, we find a family of $2^{\Omega(n \log(\alpha(n)))}$ permutations with VC-dimension 3, which gives a negative answer to a question of Raz. (The function $\log^\star(n)$ is the inverse of the tower function and $\alpha(n)$ is the inverse of the Ackermann function.)

We also study a related extremal problem of determining the maximum number of 1-entries in an $n \times n$ $(0,1)$-matrix with no $k$-tuple of columns containing all $k$-permutation matrices. From a result of Raz, it is known that this number grows linearly in $n$ if $k \leq 3$. For any fixed $k \geq 4$, we show bounds $\Omega(n\alpha(n))$ and $O(n2^{\alpha^{O(1)}(n)})$. The upper bound is an easy corollary of Klazar's result on generalized Davenport-Schinzel sequences. The lower bound follows from the result of Füredi and Hajnal on forbidden $(0,1)$-matrices, which is based on a construction of Davenport-Schinzel sequences by Hart and Sharir.

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