Some general results for even-Wilf-equivalence

Andrew M. Baxter[1] Rutgers University

For a set of (classical) patterns B, let $\mathcal{S}_n(B)$ be the permutations in the symmetric group $\mathcal{S}_n$ that avoid every pattern in B. Let $\mathcal{E}_n$ and $\mathcal{O}_n$ denote the sets of even and odd, respectively, permutations in $\mathcal{S}_n$; let $\mathcal{E}_n(B) = \mathcal{S}_n(B) \cap \mathcal{E}_n$ and $\mathcal{O}_n(B) = \mathcal{S}_n(B) \cap \mathcal{O}_n$, and let $E_n(B) = |\mathcal{E}_n(B)|$ and $O_n(B) = |\mathcal{O}_n(B)|$. We say that two sets of classical patterns B and C are *even-Wilf equivalent* (or *$\mathcal{E}_n$-Wilf equivalent*) if, for every $n \geq 1$, $E_n(B) = E_n(C)$; we then write $B \sim_{\mathcal{E}_n} C$ (and we write $B \sim_{\mathcal{S}_n} C$ for classical Wilf equivalence). Here we consider the problem of determining the even-Wilf equivalences between the singleton subsets of $\mathcal{S}_n$. In particular, we prove general even-Wilf equivalences that parallel previously known families of Wilf equivalences and involution-Wilf equivalences.

Initial work was done by Simion and Schmidt for $B \subseteq \mathcal{S}_3$; their work implies that $123 \sim_{\mathcal{E}_n} 231 \sim_{\mathcal{E}_n} 312$ and $132 \sim_{\mathcal{E}_n} 213 \sim_{\mathcal{E}_n} 321$. However, it is not the case in general that $\sigma \sim_{\mathcal{E}_n} \tau$ whenever $\sigma \sim_{\mathcal{S}_n} \tau$ and σ and τ have the same parity. This is demonstrated by the two even patterns 1234 and 4321: $1234 \sim_{\mathcal{S}_n} 4321$, but $E_6(1234) = 258$ and $E_6(4321) = 255$.

For patterns $\sigma \in \mathcal{S}_k$ and $\tau \in \mathcal{S}_\ell$, let $\sigma \oplus \tau \in \mathcal{S}_{k+\ell}$ be their direct sum $\sigma_1\sigma_2 \ldots \sigma_k(k + \tau_1)(k + \tau_2) \ldots (k + \tau_\ell)$. Backelin, West, and Xin showed that $k \ldots 21 \oplus \sigma \sim_{\mathcal{S}_n} 12 \ldots k \oplus \sigma$. One proof of this used an intermediate result that $k(k-1) \ldots 21 \oplus \sigma \sim_{\mathcal{S}_n} (k-1)(k-2) \ldots 21k \oplus \sigma$. We show that this result extends to $k(k-1) \ldots 21 \oplus \sigma \sim_{\mathcal{E}_n} (k-1)(k-2) \ldots 21k \oplus \sigma$ whenever k is odd. This allows us to classify all patterns $\sigma \in \mathcal{S}_4$ according to even-Wilf equivalence, and it shows a parallel between even-Wilf equivalence and both classical Wilf equivalence and involution-Wilf equivalence. However, because even-Wilf equivalence is not respected by all the symmetries of the square, illustrated above by $1234 \not\sim_{\mathcal{E}_n} 4321$, this result does not complete the classification of $\sigma \in \mathcal{S}_5$ according to even-Wilf equivalence.

This is joint work with Aaron D. Jaggard.

[1] Supported in part by NSA grant H98230-09-1-0014.