Math 8, Winter 2005

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Functions of several variables

A *function of two variables*, f, is a rule that assigns to each vector $< x, y > \in D \subset \mathbb{R}^2$ a real number denoted by $f(x, y)$. The set D is called the domain of f and its range is the set of values that f takes on, i.e. $\{t\}$ where $f(x, y) = t$ for some $< x, y > \in D$.

Examples:

- $f(x, y) = x^2 + y^2$
- $f(x, y) = \sin(xy)$
- $f(x, y) = \sqrt{(}1 - x^2 - y^2)$

Plotting

contour plots

For $f(x, y)$ graph $f(x, y) = k$ for different values of k and put together in a graph.

Example: $f(x, y) = x^2 + y^2$

- For $k > 0$
$$x^2 + y^2 = k$$
is a circle of radius $\sqrt{k}$.
- For $k < 0$
$$x^2 + y^2 = k$$
has no solutions.
- For $k = 0$
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consists of the single point $(0, 0)$.

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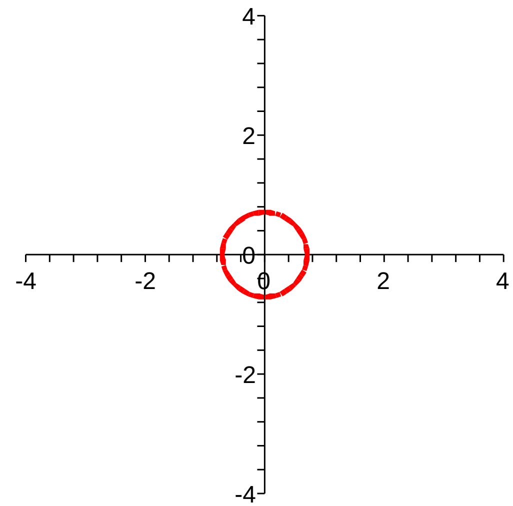

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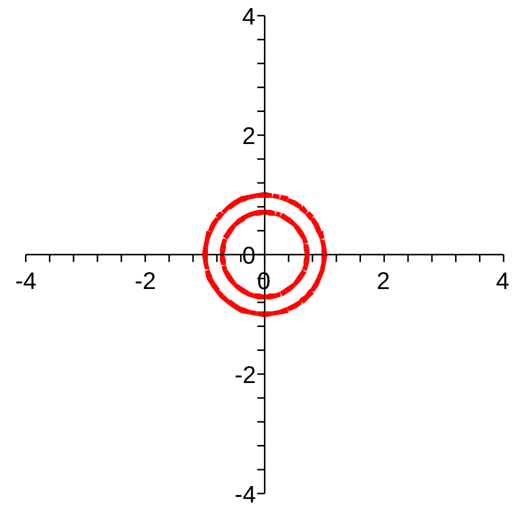

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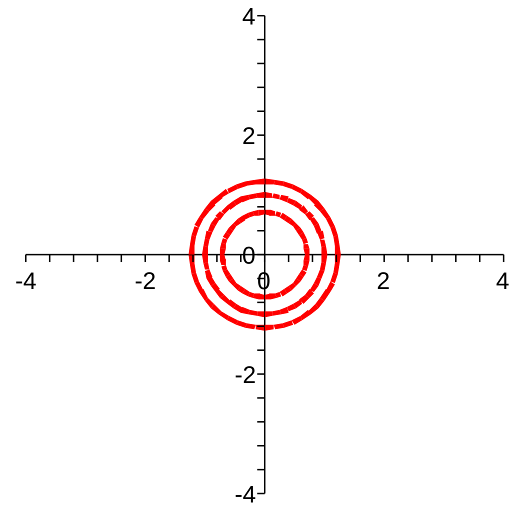

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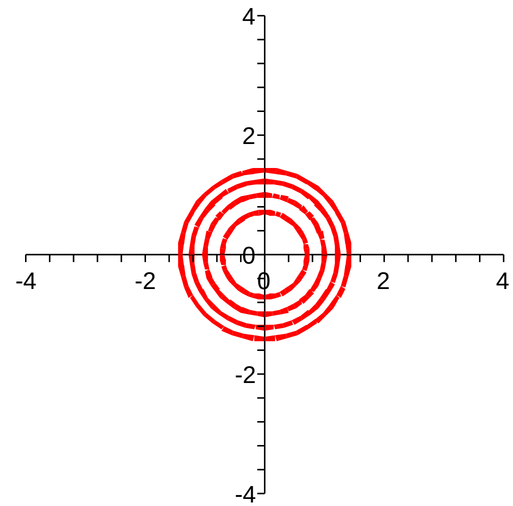

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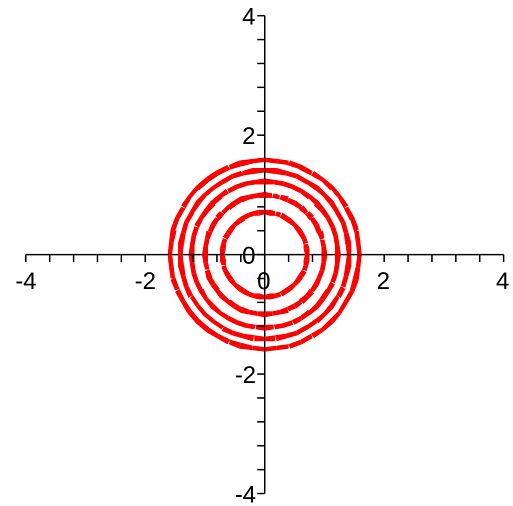

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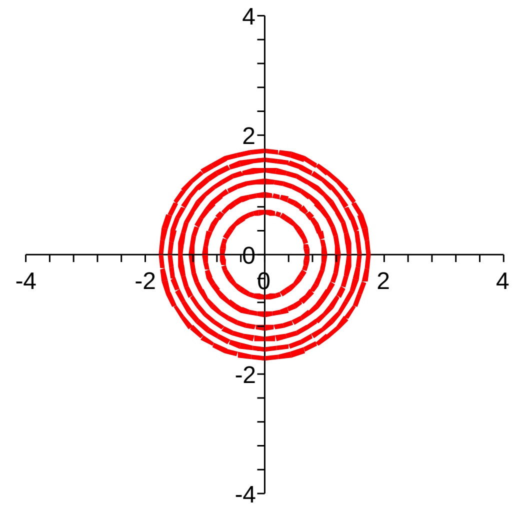

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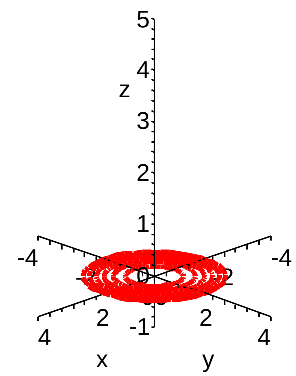

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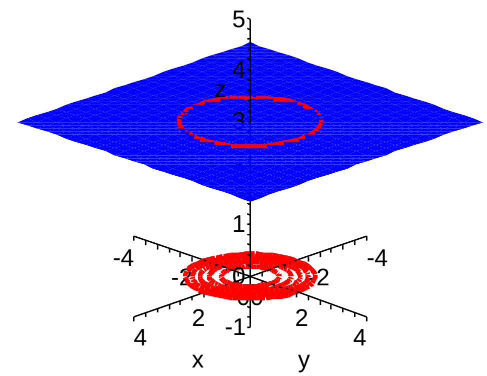

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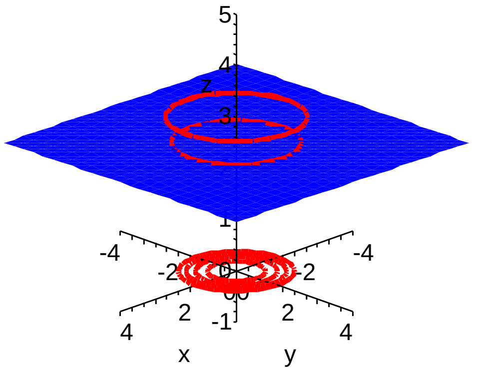

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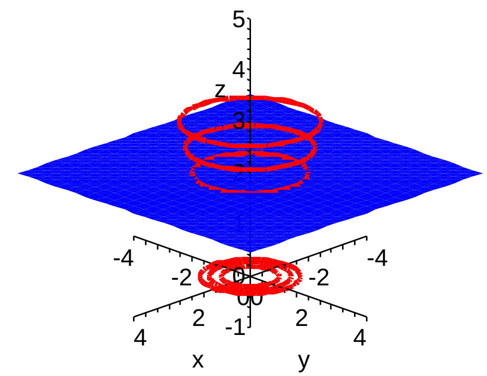

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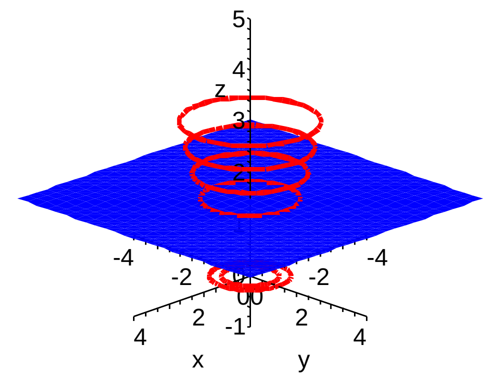

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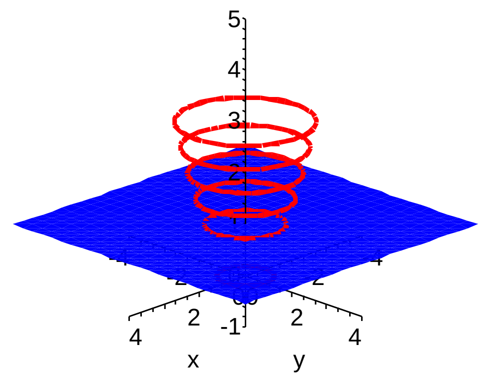

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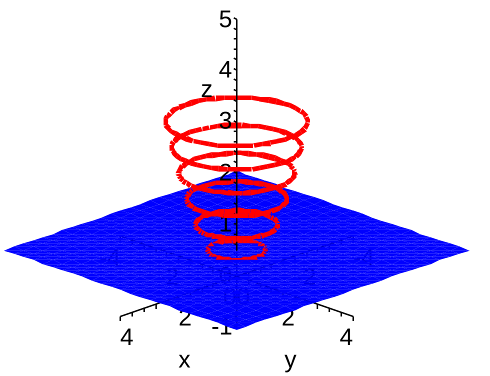

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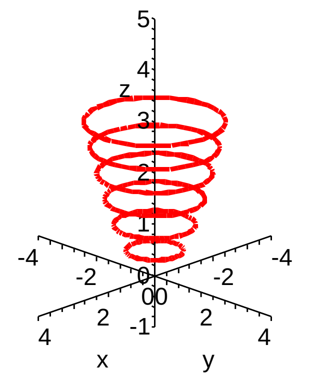

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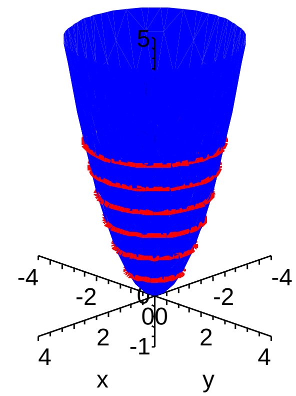

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Limits

Limits in more than one variable are much harder than in a single variable.

Let f be a function of two variables. Then,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, given an $\varepsilon > 0$ there is a $\delta > 0$ so that if the distance between (x, y) and (x_0, y_0) is less than δ then

$$|f(x,y) - L| < \varepsilon$$

Limits

Proving a limit exists is difficult, but sometimes showing one does not exists is easier.

- Look at the function restricted to different lines through (x_0, y_0).
- If the limit along one line is different from the limit along a different line, then the limit does not exist.
- Example:

$$f(x,y) = \frac{xy^2}{x^2+y^2}$$

Continuity

A function of two variable f is *continuous* at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Examples:

- Polynomials
- Rational functions: discontinuities when the denominator is zero