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So far, we have considered only functions of one variable:

$$f : \mathbb{R} \to \mathbb{R}$$

e.g. $f(x) = \sin(x^2) + 3x - 1$. We now move to considering functions that take more than one variable as input or that return more than one number as output. To understand this, we introduce the notation to describe points in the plane and in space:

Definition 1. *A point in $\mathbb{R}^2$ is a pair of numbers (x, y) that describe a point in the plane. A point in $\mathbb{R}^3$ is a triple of numbers (x, y, z) that describe a point in space.*

It will be useful to look at point in a different way as well, as vectors:

Definition 2. *A vector $\vec{v} =< x, y >$ in $\mathbb{R}^2$ denotes a displacement in the plane while a vector $\vec{w} =< x, y, z >$ is $\mathbb{R}^3$ denotes a displacement in space.*

Vectors

Vectors are determined by two quantities: *magnitude* and *direction*.

The magnitude of a vector is given by the formula:

$$|\vec{v}| = |<a,b,c>| = \sqrt{a^2+b^2+c^2}$$

The direction of a vector $\vec{v}$ is the *unit vector* created from $\vec{v}$:

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

Vector algebra

We can add vectors by adding their components. If $\vec{v} =< a, b, c >$ and $\vec{w} =< d, e, f >$ then

$$\vec{v} + \vec{w} =< a + d, b + e, c + f >$$

and

$$\vec{v} - \vec{w} =< a - d, b - e, c - f >$$

We can also multiply them by constants - if M is a constant then

$$M\vec{v} =< Ma, Mb, Mc >$$

Note: The vector $\vec{v} - \vec{w}$ is a copy of the vector pointing from the *point* (d, e, f) to the *point* (a, b, c).

Distances between points

Given two points $P_1 = (a, b, c)$ and $P_2 = (\alpha, \beta, \gamma)$ in $\mathbb{R}^3$, the distance between them is given by

$$|P_1P_2| - \sqrt{(a-\alpha)^2 + (b-\beta)^2 + (c-\gamma)^2}$$

Note that this is the same as the magnitude of the vector $\vec{v} = \overrightarrow{P_1P_2}$.

Common objects

Spheres:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Coordinate Planes:

xy-plane: $z = 0$ yz-plane: $x = 0$ xz-plane: $y = 0$

Projections to planes: We can always project points to the coordinate planes using “forgetful” functions:

$$P^{xy}((a,b,c)) = (a,b,0)$$
$$P^{yz}((a,b,c)) = (0,b,c)$$
$$P^{xz}((a,b,c)) = (a,0,c)$$

Applications

The standard basis vectors:

$$\vec{i} =< 1, 0, 0 > \quad \vec{j} =< 0, 1, 0 > \quad \vec{k} =< 0, 0, 1 >$$

Any vector $\vec{v} =< a, b, c >$ can be written in terms of the standard basis vectors:

$$\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$$

Question: What is the distance from a point (a, b, c) to the sphere

$$x^2 + y^2 + z^2 = 4$$