Math 8, Winter 2005

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2/4/05

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Power Series

There are several ways to make new power series out of old ones:
Integration:

$$\int \sum_{n=0}^{\infty} c_n (x-a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$$

Differentiation:

$$\frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n \, dx = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

Substitution: for example, let $x = \alpha u + \beta$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n (\alpha u + \beta - a)^n = \sum_{n=0}^{\infty} c_n \alpha^n \left(u + \frac{\beta - a}{\alpha} \right)^n$$

Power Series

Integration and Differentiation do not change the radius of convergence, but substitution may!

Suppose $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence R, i.e.

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = R$$

Then, after substituting $x = \alpha u + \beta$, we have the series:

$$\sum_{n=0}^{\infty} c_n \alpha^n \left(u + \frac{\beta - a}{\alpha} \right)^n$$

Performing the ratio test yields:

$$\lim_{n\to\infty} \left| \frac{c_{n+1}\alpha^{n+1}\left(u + \frac{\beta-a}{\alpha}\right)^{n+1}}{c_n \alpha^n \left(u + \frac{\beta-a}{\alpha}\right)^n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{c_{n+1}\alpha}{c_n} \right| \left| \left(u + \frac{\beta - a}{\alpha} \right) \right| < 1$$

Substitution (con't)

$$\lim_{n\to\infty}\left|\frac{c_{n+1}\alpha}{c_n}\right|\left|\left(u-\frac{\beta-a}{\alpha}\right)\right| < 1$$

$$\left|\left(u-\frac{\beta-a}{\alpha}\right)\right| < \frac{1}{\alpha}\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

$$= \frac{R}{\alpha}$$

So the radius of convergence of the new series after substitution is $\frac{R}{\alpha}$.

Power series representations

Power series define functions on their intervals of convergence. Can we find a different description for the function?

$$\sum_{n=0}^{\infty} c_n (x-a)^n = f(x)$$

Or, given a function, can we find a power series representation?

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Building representations

Example: geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Use this to find a power series representation for

- $$f(x) = \frac{1}{(1-x)^2}$$
- $$f(x) = \ln(1-x)$$
- $$f(x) = \frac{1}{1+2x^2}$$

Building representations

What about a function like

$$f(x) = \sin(x)$$

Idea:

- Goal: a power series representation about a
- The m^{th} partial sum of a series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is a polynomial of degree m:

$$s_m = c_0 + c_1(x-a) + c2(x-a)^2 + \cdots + c_m(x-a)^m$$

- Find polynomials that closely match the function, $f(x)$, near $x = a$.

$m = 0$ and $m = 1$

$\boxed{m = 0}$

$s_0 = c_0$ so simply pick $c_0 = f(a)$.

$\boxed{m = 1}$

$s_1 = c_0 + c_1(x - a) = f(a) + c_1(x - a)$

Find c_1 so that $f(x) - (f(a) + c_1(x - a))$ is as small as possible near $x = a$:

$$f(x) - (f(a) + c_1(x - a)) = (f(x) - f(a)) + c_1(x - a)$$

Dividing by $(x - a)$ gives:

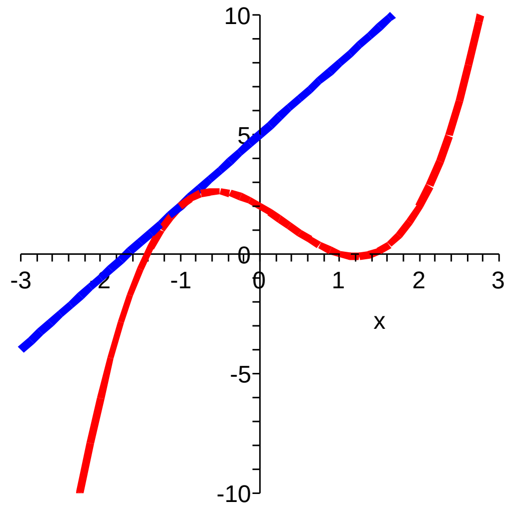
$$\frac{(f(x) - f(a))}{x - a} + c_1 \sim f'(a) + c_1$$

when x is close to a. So pick $c_1 = f'(a)$.

$m = 1$ geometrically

Using these values, what is $s_1 = f(a) + f'(a)(x - a)$?

$m = 1$ geometrically



Using these values, what is $s_1 = f(a) + f'(a)(x - a)$?

It is just the tangent line to f at $x = a$!