Answer ALL questions. Unless instructed otherwise, you should show ALL your work and simplify your final answer as much as possible. Please box your final answer to each part.

Problem 1: [12 pts]

(a) Compute the Taylor series for $f(x) = \cos x$ with center $a = \pi/2$.

Solution:

Computing derivatives yields $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, $f^{(5)}(x) = \cos x$. Henceforth the derivatives cycle round following that pattern. Therefore at $x = \pi/2$ we have the pattern of Taylor coefficients $0, -1, 0, 1, 0, -1, 0, 1, \ldots$. Thus the Taylor series is

$$T(x) = -(x - \pi/2) + \frac{1}{6}(x - \pi/2)^3 - \frac{1}{5!}(x - \pi/2)^5 + \ldots$$
$$= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{(2k+1)!}(x - \pi/2)^{2k+1}.$$

(b) Use your answer to part (a) to express $\cos\left(\frac{\pi}{2} + \frac{1}{10}\right)$ as a fraction, accurate to within $1/100000$ ($= 10^{-5}$).

Solution:

Method 1: The absolute values of all derivatives of $f(x)$ are bounded above by 1. Thus the Taylor remainder formula with $d = \frac{1}{10}$ yields

$$|R_n(\pi/2 + 1/10)| \leq \frac{1}{(n+1)!}\left|\frac{1}{10}\right|^{n+1}.$$

We need the error to be smaller than $1/100000$ which since $(3+1)! = 24$ implies we must take $n \geq 3$ (Note this is actually $k \geq 1$ for the series above as the power is $2k + 1$). Therefore

$$\cos(\pi/2 + 1/10) \approx -\frac{1}{10} + \frac{1}{6}\frac{1}{10^3} = -\frac{599}{6000}$$

with error $< 1/100000$.

Method 2: Compute $\cos(\pi/2 + 1/10) = \sum_{k=0}^{infty} (-1)^{k+1}\frac{1}{(2k+1)!}\left(\frac{1}{10}\right)^{2k+1}$. This is an alternating series. Therefore by the alternating series estimation theorem, to estimate the true value to the desired accuracy, we only need to add terms until the next one is less than 10^{-5}. When $k = 2$, the b_2 term is $10^{-5}/5! < 10^{-5}$. So we only need the $k = 0$ and $k = 1$ terms. Again

$$\cos(\pi/2 + 1/10) \approx -\frac{1}{10} + \frac{1}{6}\frac{1}{10^3} = -\frac{599}{6000}$$

Problem 2: [13 pts]   Three points are given by $A(1, 0, -1)$, $B(2, 2, -2)$ and $C(-1, 3, 0)$.

(a) Find the scalar projection of $\vec{BA}$ onto $\vec{BC}$.

Solution:

Now $\vec{BA} =< -1, -2, 1 >$ and $\vec{BC} =< -3, 1, 2 >$. Therefore

$$\text{comp}_{\vec{BC}} \vec{AB} = \frac{\vec{AB} \cdot \vec{BC}}{|\vec{BC}|} = \frac{3 - 2 + 2}{\sqrt{14}} = \frac{3}{\sqrt{14}}.$$

(b) Find the area of the triangle $\triangle ABC$.

Solution:

The triangle is spanned by $\vec{BA}$ and $\vec{BC}$, thus

$$\text{Area} = \frac{1}{2} \left| \vec{BA} \times \vec{BC} \right| = \frac{1}{2} |< -4 - 1, -3 + 4, -2 - 6 >|$$
$$= \frac{1}{2} |< -5, -1, -7 >| = \frac{\sqrt{75}}{2}.$$

(c) Find the angle $\angle ABC$. (i.e. the angle at B.) You should leave your answer in the form of the arccos of a number.

Solution:

$\vec{BA} =< -1, -2, 1 >$, $\vec{BC} =< -3, 1, 2 >$. Thus

$$\cos \theta = \frac{\vec{BA} \cdot \vec{BC}}{|BA| |BC|} = \frac{3 - 2 + 2}{\sqrt{6}\sqrt{14}} = \frac{3}{\sqrt{14}\sqrt{6}}.$$

Thus $\theta = \arccos \left(\frac{3}{\sqrt{14}\sqrt{6}} \right)$.

(d) Find a vector equation of the line containing A and B.

Solution:

$A(1, 0, -1)$ is a point on the line and $\vec{AB} =< 1, 2, -1 >$ is a vector pointing along the line. Thus a vector equation for the line is given by

$$\vec{r}(t) =< 1, 0, -1 > +t < 1, 2, -1 > .$$