Math 8 Final Exam Practice Problems - Partial Answers

Disclaimer: These problems are meant to help you practice for the exam on the material that has been covered since the second midterm exam. You already have the two midterm exams and two previous practice exams to help study the other material in the course. These are in no way a comprehensive set of problems - you may well see completely different types of problems on the final exam.

1. Find a vector tangent to the curve of intersection of the two surfaces $z = x^2 - y^2$ and $xyz + 30 = 0$ at the point $(-3, 2, 5)$.

   Answer: Let $f(x, y, z) = x^2 - y^2 - z$ and $g(x, y, z) = xyz + 30$. The vector in question is $\vec{T} = \nabla f(-3, 2, 5) \times \nabla g(-3, 2, 5) = \langle 9, -46, 130 \rangle$.

2. Let $f(x, y, z) = ye^{-x^2} \sin(z)$. Find the equation of the tangent plane to the level surface of f at the point $(0, 1, \pi/3)$.

   Answer: The normal to the tangent plane is $\vec{n} = \nabla f(0, 1, \pi/3) = \langle 0, \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$. An equation for the tangent plane is $\sqrt{3}(y - 1) + (z - \frac{\pi}{3}) = 0$.

3. Let $w = e^{xy}, x = f(s, t), y = g(s, t)$. Suppose also that

$$f(0, 2) = 1, f_s(0, 2) = 3, f_t(0, 2) = 5, g(0, 2) = 2, g_s(0, 2) = -1, g_t(0, 2) = 0.$$

   a.) Find $\frac{\partial w}{\partial s}$ at the point $(s, t) = (0, 2)$.

   Answer: When $(s, t) = (0, 2)$ we have $(x, y) = (1, 2)$. By the Chain Rule,

$$\frac{\partial w}{\partial s}(0, 2) = \frac{\partial w}{\partial x}(1, 2) f_s(0, 2) + \frac{\partial w}{\partial y}(1, 2) g_s(0, 2) = 5e^2.$$

   b.) Calculate the directional derivative of f in the direction given by $\vec{v} = \langle 3, 4 \rangle$.

   Answer: Let $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$. Then the required directional derivative is

$$D_{\vec{u}} f(0, 2) = f_s(0, 2)\frac{3}{5} + f_t(0, 2)\frac{4}{5} = \frac{29}{5}.$$

4. Consider the sphere $x^2 + y^2 + z^2 = r^2$ and the elliptic cone $z^2 = a^2x^2 + b^2y^2$, where a, b, r are constants. Show that at every point of their intersection, the corresponding tangent planes are orthogonal.

   Answer: Let $f(x, y, z) = x^2 + y^2 + z^2 = r^2$ and $g(x, y, z) = a^2x^2 + b^2y^2 - z^2 = 0$. We need only show that $\nabla f(x, y, z) \perp \nabla g(x, y, z)$ at each point (x, y, z) in the intersection. We compute:

$$\nabla f(x, y, z) \cdot \nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle \cdot \langle 2a^2x, 2b^2y, -2z \rangle = 4(a^2x^2 + b^2y^2 - z^2) = 0$$

   and so they are orthogonal as desired.

5. A ball is placed at the point $(1,2,3)$ on the surface $z = y^2 - x^2$. Give the direction in the xy-plane that the ball will start to roll.

   Answer: Let $z = f(x,y) = y^2 - x^2$. The ball will roll in the direction given by $\vec{v} = -\nabla f(1,2) = \langle 2, -4 \rangle$.

6. Let $f(x,y) = x^4 + y^4 + x^2 - y^2$.
   a.) Find and classify all the critical points of f.

   Answer: The function f has a saddle point at the origin $(0,0)$ and local (absolute) minima at $(0, \pm\frac{\sqrt{2}}{2})$.

   b.) Use the method of Lagrange multipliers to find the largest and smallest values of f on the circle $x^2 + y^2 = 4$.

   Answer: The maximum value of f is 20 at the points $(\pm 2, 0)$. The minimum value of f is 5 at the points $(\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{5}{2}})$.

7. Find the extreme values of the function $f(x,y) = x^2 y e^{-(x+y)}$ on the triangular region given by $x \geq 0, y \geq 0$ and $x + y \leq 4$.

   Answer: The maximum value of f is $4/e^3$ and it occurs at $(2,1)$. The minimum value of f is zero and occurs at all points $(x,0)$ and $(0,y)$ on the two perpendicular boundary segments of the triangle.

8. The temperature at the point (x,y,z) is given by $T(x,y,z) = xy^2z$. Find the direction of maximum increase in temperature at the point $(1,-2,3)$. If you move on a straight line so that your velocity as you pass through the point $(1,-2,3)$ is $\vec{v} = \langle 2,1,2 \rangle$ then what is the rate of temperature increase as you pass through $(1,-2,3)$?

   Answer: The direction of maximum increase in temperature at the point $(1,-2,3)$ is given by $\nabla T(1,-2,3) = \langle 12, -12, 4 \rangle$. Let $\vec{u} = \langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$. As you pass through $(1,-2,3)$ with velocity $\vec{v} = \langle 2,1,2 \rangle$ the rate of temp increase is

$$D_{\vec{u}}T(1,-2,3) = \langle 12, -12, 4 \rangle \cdot \vec{u} = \frac{24}{3} - \frac{12}{3} + \frac{8}{3} = \frac{20}{3}.$$

9. Find the absolute extreme values of the function $f(x,y) = \frac{3}{2}x^2 + x + y^2 - 2$ on the closed disk $x^2 + y^2 \leq 4$.

   Answer: The minimum value of f is $-\frac{13}{6}$ and occurs at $(-\frac{1}{3}, 0)$ and the maximum value is 2 and occurs at $(0, \pm 2)$.