Math 8 Practice Exam 2 Problem Answers

1. Find the equations of the following lines and planes:
(a) the plane through the points $(1, 1, 0)$, $(1, -4, 2)$ and $(-2, 3, 1)$.
(b) the plane through the origin and parallel to the plane $5x - 2y + z = 15$.
(c) the line through the point $(1, 2, 3)$ and perpendicular to the plane $x - 3y + 5z = 1$.

Answers: (a) $3x + 2y + 5z = 5$ (b) $5x - 2y + z = 0$ (c) $\langle x, y, z\rangle = \langle 1+t, 2-3t, 3+5t\rangle$

2. (a) Find the area of the triangle with vertices $(1, -2, 5)$, $(1, 3, 0)$ and $(1, 0, 1)$.
(b) Find the volume of the parallelepiped determined by the vectors $\vec{\mathbf{u}} = \langle 1, 2, 3\rangle$, $\vec{\mathbf{v}} = \langle 2, 0, 1\rangle$ and $\vec{\mathbf{w}} = \langle 3, 0, 4\rangle$.

Answers: (a) $A = \frac{1}{2}|\vec{P_0P_1} \times \vec{P_0P_2}| = \frac{1}{2}|\langle -10, 0, 0\rangle| = 5$ (b) $V = |\vec{\mathbf{u}} \bullet (\vec{\mathbf{v}} \times \vec{\mathbf{w}})| = |-10| = 10$.

3. Consider the pair of planes

$$x + y + 7z = 9 \text{ and } (2, -1, 5) \bullet (x, y, z) = 6.$$

(a) Find a vector representation of the line of intersection of these planes.
(b) Find the cosine of the angle between these planes.

Answers: (a) $\vec{\mathbf{r}}(t) = \langle 5 - 4t, 4 - 3t, t\rangle$ (b) $\cos(\theta) = |\vec{\mathbf{n}}_1 \bullet \vec{\mathbf{n}}_2|/(|\vec{\mathbf{n}}_1||\vec{\mathbf{n}}_2|) = \frac{12}{\sqrt{170}}$.

4. Consider the curve given parametrically by $g(t) = (t^2, \frac{1}{3}t^3),\ -2 \le t \le 2$.
(a) Find the arc length of the curve from $t = 0$ to $t = 2$.
(b) Find a vector representation of the tangent line to the curve at $t = 1$.
(c) Suppose a particle is moving along the curve and then flies off at time $t = 1$ second, travelling at a constant speed along the tangent line. Find the position of the particle two seconds later.

Answers: (a) $L = \int_0^2 |g'(t)|\, dt = \int_0^2 \sqrt{4t^2 + t^4}\, dt = \int_0^2 t\sqrt{4 + t^2}\, dt = \frac{1}{3}(16\sqrt{2} - 8)$. (b) $\vec{\mathbf{r}}(t) = g(1) + tg'(1) = \langle 1 + 2t, \frac{1}{3} + t\rangle$. (c) $\vec{\mathbf{r}}(2) = \langle 5, 2\frac{1}{3}\rangle$.

5. (a) Find the projection of the vector $(1, 3, -2)$ onto the vector $(1, 1, 1)$.
(b) Write $(1, 3, -2)$ as the sum of a vector parallel to $(1, 1, 1)$ and a vector perpendicular to $(1, 1, 1)$.

Answers: (a) $\langle \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\rangle$ (b) $\langle 1, 3, -2\rangle = \langle \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\rangle + \langle \frac{1}{3}, \frac{7}{3}, -\frac{8}{3}\rangle$.

6. The curves $\vec{\mathbf{r}}_1(t) = \langle t^3, t^2, t\rangle$ and $\vec{\mathbf{r}}_2(t) = \langle \sin(t) + 1, e^t, \cos(t)\rangle$ intersect at the point $(1, 1, 1)$. Find the equation of the plane containing both tangent lines at $(1, 1, 1)$.

Answers: Normal $\vec{\mathbf{n}} = \vec{\mathbf{r}}_1'(1) \times \vec{\mathbf{r}}_2'(0) = \langle -1, 1, 1\rangle$. The plane is $-x + y + z = 1$.

7. (a) Find the center and radius of the sphere $2x^2 + 2y^2 + 2z^2 + 6x - 8z + 2 = 0$.
(b) Find the arc length of the curve $\vec{\mathbf{r}}(t) = \langle t, \frac{t^3}{3} + 4, \frac{\sqrt{2}}{2}t^2\rangle$ from $t = 0$ to $t = 1$.

Answers: (a) Center: $(-\frac{3}{2}, 0, 2)$, Radius: $r = \frac{\sqrt{21}}{2}$ (b) $L = \int_0^1 \sqrt{1 + 2t^2 + t^4}dt = \int_0^1 (1 + t^2)dt = \frac{4}{3}$.

8. Are the lines $\frac{x-1}{4} = \frac{y-2}{-4} = \frac{z-3}{6}$ and $\vec{\mathbf{r}}(t) = \langle 3+t, 2t, 6+3t\rangle$ skew, parallel, or intersecting?

   Answers: The lines intersect at $(3, 0, 6)$.

9. A helicopter is to fly directly from the helipad at the origin toward the point $(1, 1, 1)$ at a speed of 60 feet/sec (and continue flying in that direction.)
   (a) Find a unit vector $\vec{\mathbf{u}}$ in the direction of motion.
   (b) Using part (a), what is the position of the helicopter after t seconds? after 10 seconds?

   Answers: (a) $\vec{\mathbf{u}} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$. (b) $\vec{\mathbf{r}}(t) = 20\sqrt{3}t\langle 1, 1, 1\rangle$. After 10s $\vec{\mathbf{r}}(10) = \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3}\rangle$.

10. Consider the function $f(x, y) = \sqrt{1 - \ln(x^2 + y^2)}$
    (a) Find the domain of f.
    (b) Compute the limit as $(x, y) \to (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ of $f(x, y)$.
    (c) Compute the partial derivatives f_x and f_y.

    Answers: (a) Domain $D = \{(x, y) \in \mathbf{R}^2 \mid 0 < x^2 + y^2 \leq e\}$

    (b) The limit is $\lim_{(x,y)\to(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})} \sqrt{1 - \ln(x^2 + y^2)} = 1$.

    (c) The partial derivatives are:

$$f_x(x, y) = \frac{-x}{(x^2 + y^2)\sqrt{1 - \ln(x^2 + y^2)}}$$

$$f_y(x, y) = \frac{-y}{(x^2 + y^2)\sqrt{1 - \ln(x^2 + y^2)}}$$